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Mathematical Analysis of Optimal Control Theory on Underweight

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Authors' contributions

This work was carried out in collaboration between all authors. Author NHS discussed concept and then authors FAT and BMY designed it. Authors FAT and BMY anchored the field study, gathered the initial data and performed data analysis. All authors read and approved the final manuscript.

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ABSTRACT

To achieve increase in the number of healthy people in the society, proper information and understanding about health care issues is required. This will bring about improvement and betterment of the health of the population, especially to the girls and women of the society who are welcoming the new generation. Females need to take an adequate amount of healthy and nutritive food in their daily life for themselves as well as for the development and well-being of the new generation so as to prevent malnutrition and underweight which may cause deficiency in the various parts and organs of the body. In this paper, we want to study the transmission of malnutrition and underweight individuals in the society. The strategy in terms of healthy life campaign and treatment for healthy individual in the society is focussed. The model is supported with numerical simulation.

Keywords: Transmission; malnutrition; underweight; local and global stability in terms of control; simulation.

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1. INTRODUCTION

The importance of health can be viewed from different perspectives. When we are talking of health, it is not just about a healthy body but also sound about mental health. Good health can be described as the condition where both our body as well as our mind are functioning properly. Every individuals' role in life, requires a good health to perform effectively. Better health is central to human population and well- being. A healthy diet is vital for maintaining health and fitness. [1] argued that access to adequate nutritious food is a basic human right. Timely nutrition will help faster recovery on malnutrition as the world drives for Sustainable Development Goals. The main causes behind poor health conditions are diseases, injury, mental stress, lack of hygiene, unhealthy life style, poverty and improper diet. The above factor forces people to suffer from malnutrition and as a result underweight. The recent search shows that 15.2% of the Indian population are undernourished [2]. [3] focused on policy developers and community thinkers at national and international level have expressed their anxiety over the critical state of undernutrition in the country.

Malnutrition is a most serious and common health problem that occurs when a person's diet does not contain the proper amounts of nutrients. This may occur due to problem of absorbing nutrients from food also. Physical factors and Medical condition can lead to malnutrition. Also, social factors like poverty, alcohol or drug dependency etc. may lead to malnutrition. Symptoms of malnutrition includes weak muscles, feeling tired all the time, low mood, an increase in illness orientation [4]. Inadequate nutrition is especially critical for women because it not only affects their own health but also the health of their children. Its sign in children can include failure of natural growth at the expected rate.

Malnutrition in adult leads to underweight. It is a term describing a person whose body weight is considered too low, compared to prescribed medical fraternity. It refers to those adults with a Body Mass Index (BMI) of under 8.5 or a weight 15% to 20% below the normal for that age and height [5]. People who are underweight raise special concerns, it has been reported that underweight increases mortality at higher rates as compared to morbidly obese people.

Addressing women's malnutrition is important because a healthy woman can fulfil their multiples roles such as generating income, ensuring their families nutrition and having healthy children more effectively and thereby helps in socio economic development of the countries [6,7]. It has been observed that women neglect healthy nutrition which affects their health. By investing in nutrition in terms of campaign for all, we all can be benefitted. Malnutrition, and resultant underweight can be controlled. Along with it several actions like diet, exercise and appetite stimulants will also help to bring control towards it [8].

In this paper, a mathematical model using the application of *SEIR*-model is constructed as a system of non-linear ordinary differential equations for various compartments and its notations along with its parametric values is given in Section-2. The stability of the system such as local and global stability are discussed in Section-3.1 and 3.2 respectively. Optimal control model and Numerical Simulation are described in Section-4 and 5 respectively.

2. MATHEMATICAL MODEL

Here, we formulate a mathematical model for an individual under growing and malnutrition and then underweight using *SMU*-model. The model is developed with the notations defined in Table 1.

The dynamic of proposed problem is shown in Fig. 1.

The new born from the female susceptible of nutrients or malnutrients may or may not be 100% healthy. Here, we are dealing with the case that the female susceptible to nutrients' deficiency giving birth to malnourished boy / girl, which in excess of malnutrition undergoes underweight. The boy / girl is malnourished at the transmission rate β_B / β_G respectively from susceptible females, which in excess moves to underweight compartment at the rate α_B / α_G respectively. Then, the boy / girl who are victim of underweight, with the help of health life campaign rate (u_1), a fraction of boys and girls denoted by γ_B and γ_G, gets little recovery and moves to malnourished boy (M_B) and malnourished girl (M_G) compartment respectively. The malnourished girls get recovered at the rate η_G on providing proper treatment (u_2) in terms of healthy food and vitamins, and moves to susceptible female (S_F) compartment.

Table 1. Notations and its parametric values

Notations		Parametric values
N	Sample size	1000
S_F	Number of females susceptible to nutrients	30
M_B	Number of malnourished boys	2
M_G	Number of malnourished girls	4
U	Number of underweight individuals	1
B	New female recruitment rate	0.01
ε	Vertical Transmission to a new born population	0.001
β_B	Transmission rate of malnourished boys from susceptible female compartment	0.1
β_G	Transmission rate of malnourished girls from susceptible female compartment	0.2
μ	Mortality rate	0.1
μ_B	Natural recovery rate	0.3
α_B	Rate of boys' individuals moving from malnutrition to underweight compartment	0.01
α_G	Rate of girls' individuals moving from malnutrition to underweight compartment	0.1
γ_B	Fraction of underweight moving to M_B	0.01/4
γ_G	Fraction of underweight moving to M_G	0.01
u_1	Health Life Campaign rate (per day)	[0,1]
u_2	Treatment rate of malnourished girls (per day)	[0,1]
η_G	Recovery rate of malnourished girls	0.1

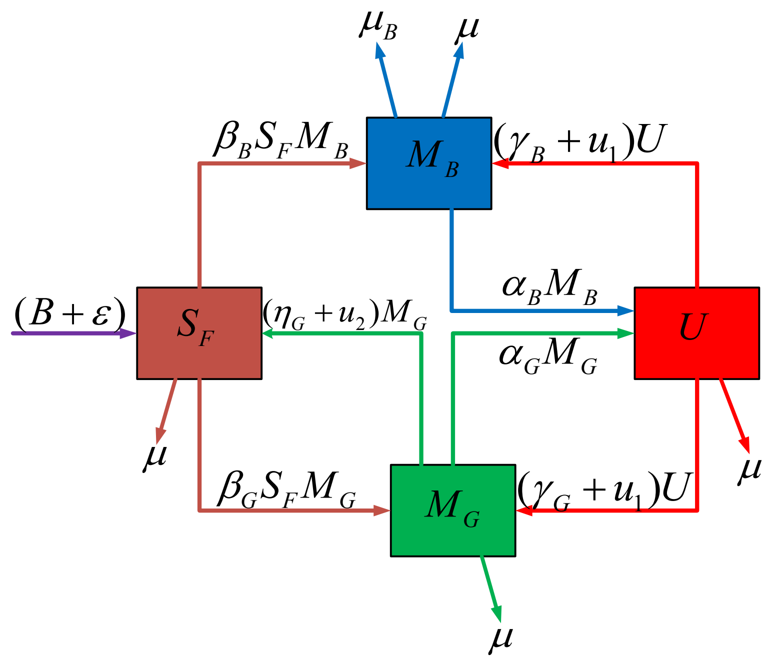

Fig. 1. Transmission of malnutrition and underweight

Now, from the above Fig. 1 we have the following set of differential equations describing the movement of boys and girls from one compartment to other.

$$\begin{aligned}
\frac{dS_F}{dt} &= (B+\varepsilon) - \beta_B S_F M_B - \beta_G S_F M_G + (\eta_G + u_2) M_G - \mu S_F \\
\frac{dM_B}{dt} &= \beta_B S_F M_B - \mu_B M_B - \alpha_B M_B + (\gamma_B + u_1) U - \mu M_B \\
\frac{dM_G}{dt} &= \beta_G S_F M_G - \alpha_G M_G + (\gamma_G + u_1) U - (\eta_G + u_2) M_G - \mu M_G \\
\frac{dU}{dt} &= \alpha_B M_B + \alpha_G M_G - (\gamma_B + u_1) U - (\gamma_G + u_1) U - \mu U
\end{aligned} \qquad (1)$$

With $S_F + M_B + M_G + U = N$ and $S_F > 0; M_B, M_G \geq 0; U \geq 0$

Adding all the above system of differential equations we get,

$$\frac{d}{dt}(S_F + M_B + M_G + U) = B + \varepsilon - \mu(S_F + M_B + M_G + U) - \mu_B M_B \geq 0$$

This gives, $\limsup\limits_{t\to\infty}(S_F + M_B + M_G + U) \leq \frac{B+\varepsilon}{\mu}$

Therefore, the feasible region for (1) is

$$\Lambda = \{(S_F + M_B + M_G + U) / S_F + M_B + M_G + U \leq \frac{B+\varepsilon}{\mu}, S_F > 0; M_B, M_G \geq 0; U \geq 0\}.$$

Thus, the equilibrium state of the model is $X_0 = \left(\frac{B+\varepsilon}{\mu}, 0, 0, 0\right)$

Now, the basic reproduction number R_0 is to be calculated using the next generation matrix method [9]. The next generation matrix is defined as FV^{-1} where F and V both are Jacobian matrices of $\Im$ and v evaluated with respect to malnourished boys and girls and underweight individuals in an equilibrium state.

Let $X = (M_B, M_G, U, S_F)$

Thus, $\frac{dX}{dt} = \Im(X) - v(X)$

where, $\Im(X)$ denotes the rate of new malnourished and underweight in the compartment and $v(X)$ denotes the rate of transfer of malnutrition which is given as

$$\Im(X) = \begin{bmatrix} \beta_B S_F M_B \\ \beta_G S_F M_G \\ 0 \\ 0 \end{bmatrix} \text{ and } v(X) = \begin{bmatrix} \alpha_B M_B - (\gamma_B + u_1) U + \mu_B M_B + \mu M_B \\ \alpha_G M_G - (\gamma_G + u_1) U + (\eta_G + u_2) M_G + \mu M_G \\ -\alpha_B M_B - \alpha_G M_G + (\gamma_B + u_1) U + (\gamma_G + u_1) U + \mu U \\ -(B+\varepsilon) + \beta_B S_F M_B + \beta_G S_F M_G + \mu S_F - (\eta_G + u_2) M_G \end{bmatrix}$$

Now, the derivative of $\Im$ and v at equilibrium point $X_0 = \left(\frac{B+\varepsilon}{\mu}, 0, 0, 0\right)$ gives matrices F and V of order 4×4 defined as

$$F = \left[\frac{\partial \Im_i (X_0)}{\partial X_j}\right] \qquad V = \left[\frac{\partial v_i (X_0)}{\partial X_j}\right] \text{ for } i, j = 1, 2, 3, 4$$

So,

$$F = \begin{bmatrix} \frac{\beta_B (B+\varepsilon)}{\mu} & 0 & 0 & 0 \\ 0 & \frac{\beta_G (B+\varepsilon)}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$V = \begin{bmatrix} \alpha_B + \mu_B + \mu & 0 & -(\gamma_B + u_1) & 0 \\ 0 & \alpha_G + \mu + \eta_G + u_2 & -(\gamma_G + u_1) & 0 \\ -\alpha_B & -\alpha_G & (\gamma_B + u_1) + (\gamma_G + u_1) + \mu & 0 \\ \frac{\beta_B (B+\varepsilon)}{\mu} & \frac{\beta_G (B+\varepsilon)}{\mu} - (\eta_G + u_2) & 0 & \mu \end{bmatrix}$$

where V is non-singular matrix. Hence, the basic reproduction number R_0 is the spectral radius of matrix FV^{-1} which is given by

$$R_O = \frac{\beta_G (B+\varepsilon)[(\mu_B + \mu)(\mu + 2u_1 + \gamma_B + \gamma_G) + \alpha_B(\mu + u_1 + \gamma_G)]}{\mu[(\mu_B + \mu)\{\mu(\mu + 2u_1 + u_2 + \alpha_G + \eta_G + \gamma_B + \gamma_G) + (\gamma_B + u_1)((\alpha_G + \eta_G) + u_2)) + (\gamma_G + u_1)(\eta_G + u_2)\}]}$$

3. STABILITY OF THE EQUILIBRIUM

In this section, the local and global stability of the malnutrition and underweight is to be discussed.

3.1 Local Stability

If all the eigenvalues of the Jacobian matrix of the system (1) have negative real parts, then the problem of malnutrition and underweight in the population is locally stable [10]. For this, at $X_0 = \left(\frac{B+\varepsilon}{\mu}, 0, 0, 0\right)$ the Jacobian of the system (1) has the form

$$J = \begin{bmatrix} -\mu & \frac{-\beta_B (B+\varepsilon)}{\mu} & \frac{-\beta_G (B+\varepsilon)}{\mu} + (\eta_G + u_2) & 0 \\ 0 & \frac{\beta_B (B+\varepsilon)}{\mu} - \mu_B - \mu - \alpha_B & 0 & (\gamma_B + u_1) \\ 0 & 0 & \frac{\beta_G (B+\varepsilon)}{\mu} - (\eta_G + u_2) - \alpha_G - \mu & (\gamma_G + u_1) \\ 0 & \alpha_B & \alpha_G & -(\gamma_B + 2u_1 + \gamma_G + \mu) \end{bmatrix}$$

Now using parametric values given in Table 1, we have

$$trace(J) = -4\mu + (\beta_B + \beta_G)\left(\frac{B+\varepsilon}{\mu}\right) - (\gamma_B + \gamma_G + 2u_1) - u_2 - \alpha_B - \alpha_G - \eta_G - \mu_B$$
$$= -0.88927 < 0$$

Thus, it is locally stable.

3.2 Global Stability

If $\det(I - FV^{-1}) > 0$ then a problem of malnutrition and underweight in the population is globally stable.

$$\det(I - FV^{-1}) = 1 - R_0 = 1 - 0.07 = 0.9 > 0$$

Hence, it is globally stable.

4. OPTIMAL CONTROL MODEL

The objective of the model is to minimize the number of individuals in malnourished and underweight compartments. The control functions are incorporated to do so. The objective function for the mathematical model of malnutrition and underweight in (1) along with the optimal control problem is given by

$$J(u_i, \Omega) = \int_0^T (A_1 S_F^{\,2} + A_2 M_B^{\,2} + A_3 M_G^{\,2} + A_4 U^2 + w_1 u_1^{\,2} + w_2 u_2^{\,2})dt \qquad (2)$$

where, Ω denotes set of all compartmental variables, A_1, A_2, A_3, A_4 denotes non-negative weight constants for S_F, M_B, M_G, U compartments respectively and w_1, w_2 are weight constants for control variables u_1, u_2 respectively.

As, the weight parameters w_1 and w_2 are constants of healthy life campaign rate (u_1) and treatment rate (u_2), which regularize the optimal control condition. The cost for healthy life campaign could come from advertisement, visiting hospitals, etc. and the cost for treatment comes from purchasing of healthy food containing required amounts of vitamins, minerals etc. to get cured from malnutrition and hence, underweight.

Now, we will calculate the values of control variables u_1 and u_2 from $t = 0$ to $t = T$ such that

$$J(u_1(t), u_2(t)) = \min\{J(u_i^*, \Omega) / (u_1, u_2) \in \phi\}$$

where ϕ is a smooth function on the interval $[0,1]$. Using the results of [11], the optimal controls denoted by $u_i^*, i = 1,2$ is obtained by collecting all the integrands of equation (2) using the lower bounds and upper bounds respectively for both the control variables.

Now, using the pontrygin's principle from [12] and [13], in order to minimize the cost function in (2) we construct Lagrangian function consisting of state equations and adjoint variables $A_V = (\lambda_{S_F}, \lambda_{M_B}, \lambda_{M_G}, \lambda_U)$ as

$$\begin{aligned} & L(\Omega, A_V) \\ & = A_1 S_F{}^2 + A_2 M_B{}^2 + A_3 M_G{}^2 + A_4 U^2 + w_1 u_1{}^2 + w_2 u_2{}^2 \\ & \quad + \lambda_{S_F}[(B+\in) - \beta_B S_F M_B - \beta_G S_F M_G + (\eta_G + u_2) M_G - \mu S_F] \\ & \quad + \lambda_{M_B}[\beta_B S_F M_B - \mu_B M_B - \alpha_B M_B + (\gamma_B + u_1) U - \mu M_B] \\ & \quad + \lambda_{M_G}[\beta_G S_F M_G - \alpha_G M_G + (\gamma_G + u_1) U - (\eta_G + u_2) M_G - \mu M_G] \\ & \quad + \lambda_U[\alpha_B M_B + \alpha_G M_G - (\gamma_B + u_1) U - (\gamma_G + u_1) U - \mu U] \end{aligned} \tag{3}$$

Now, the partial derivative of the Lagrangian function with respect to each variables of the compartment gives the adjoint equation variables $A_V = (\lambda_{S_F}, \lambda_{M_B}, \lambda_{M_G}, \lambda_U)$ corresponding to the system (1) which is as follows:

$$\begin{aligned} \dot{\lambda}_{S_F} &= -\frac{\partial L}{\partial S_F} \\ &= -2A_1 S_F + (\lambda_{S_F} - \lambda_{M_B})\beta_B M_B + \beta_G M_G(\lambda_{S_F} - \lambda_{M_G}) + \mu\lambda_{S_F} \\ \dot{\lambda}_{M_B} &= -\frac{\partial L}{\partial M_B} \\ &= -2A_2 M_B + (\lambda_{S_F} - \lambda_{M_B})\beta_B S_F + \alpha_B(\lambda_{M_B} - \lambda_U) + \lambda_{M_B}(\mu_B + \mu) \\ \dot{\lambda}_{M_G} &= -\frac{\partial L}{\partial M_G} \\ &= -2A_3 M_G + (\lambda_{S_F} - \lambda_{M_G})(\beta_G S_F - (\eta_G + u_2)) + \alpha_G(\lambda_{M_G} - \lambda_U) + \mu\lambda_{M_G} \\ \dot{\lambda}_U &= -\frac{\partial L}{\partial U} \\ &= -2A_4 U - (\lambda_{M_B} - \lambda_U)(\gamma_B + u_1) - (\lambda_{M_G} - \lambda_U)(\gamma_G + u_1) + \mu\lambda_U \end{aligned}$$

The necessary condition for Lagrangian function L to be optimal for controls using [14] are

$$\frac{\partial L}{\partial u_1} = 2w_1 u_1 + U(\lambda_{M_B} - \lambda_U) + U(\lambda_{M_G} - \lambda_U) = 0 \tag{4}$$

$$\frac{\partial L}{\partial u_2} = 2w_2 u_2 + M_G(\lambda_{S_F} - \lambda_{M_G}) = 0 \tag{5}$$

Solving equation (4) and (5) we get,

$$u_1 = \frac{U(2\lambda_U - \lambda_{M_B} - \lambda_{M_G})}{2w_1} \text{ and } u_2 = \frac{M_G(\lambda_{M_G} - \lambda_{S_F})}{2w_2}$$

Thus, the required optimal control condition is computed as

$$u_1{*} = \max\left(a_1, \min\left(b_1, \frac{U(2\lambda_U - \lambda_{M_B} - \lambda_{M_G})}{2w_1}\right)\right)$$

$$u_2{*} = \max\left(a_2, \min\left(b_2, \frac{M_G(\lambda_{M_G} - \lambda_{S_F})}{2w_2}\right)\right)$$

Now, in next section the optimal control is calculated numerically to support the analytical results.

5. NUMERICAL SIMULATION

In this section, we observe the transmission of individuals in respective group using the data given in Table 1.

Figs. 2(a) and 2(b) shows the effect on all the four compartments S_F, M_B, M_G and U without and with control respectively. It interprets that on providing healthy life campaign and treatment to the individuals in the society, the number of malnourished girls and underweight individuals decreases when control is not applied. When control is not applied, girls remain malnourished for a longer time and remains in underweight compartment for longer duration as compared to when precautions in terms of nutrients are given to malnourished girls and this reduces the density of underweight individuals.

In the initial phase, one does not require any campaign ($u_1(t)$) but after some time it demands exponentially awareness campaign for some time period and then it can be slower down. While the control ($u_2(t)$) in terms of offering, healthy food is must, but later on after decreasing for some time, it gets stabilized i.e. an individual enjoys healthy life.

This Fig. 4 shows that in both the cases of R_0, the number of malnourished boys' increases initially for few years then it starts decreasing with the passage of time and finally it gets stabilised in terms of complete recovery.

Fig. 5 shows that in case of malnourished girls, the society is at risk when reproduction number which is newly recruited individual is greater than 1. This may turn out to be epidemic and needs to be controlled as early as possible.

Fig. 6 shows that the individuals in the U compartment increase in a similar manner for different range of reproduction number. However, for reproduction number $R_0 > 1$ the individual in U compartment may be at risk for more health-related issues like weakness bones, anaemic, low metabolism etc.

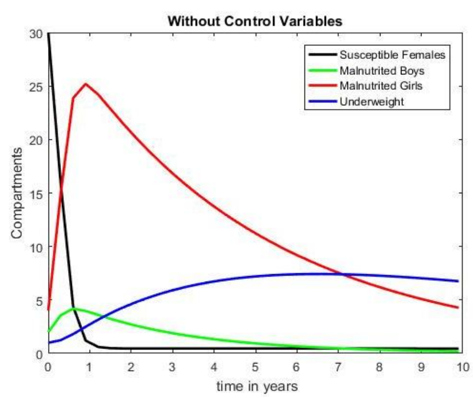

Fig. 2(a). Effect of without control on compartments

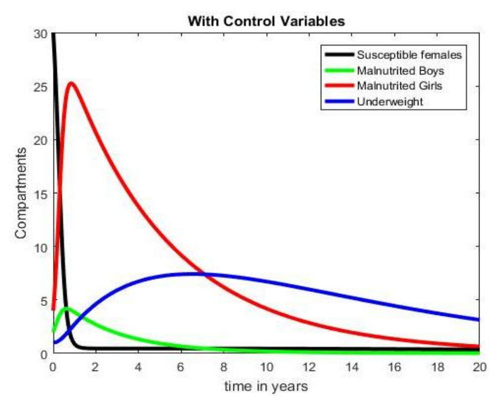

Fig. 2(b). Effect of with control on compartments

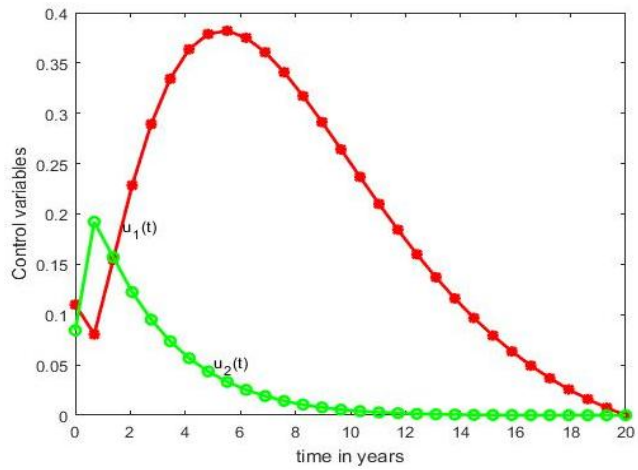

Fig. 3. Control variables with time

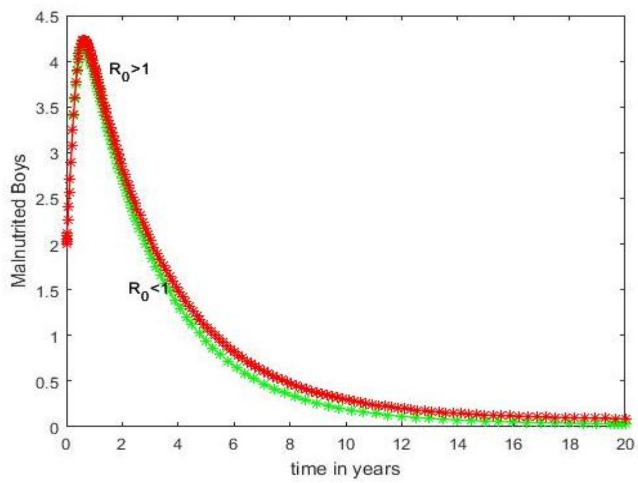

Fig. 4. Changes in malnourished boys group with R_0

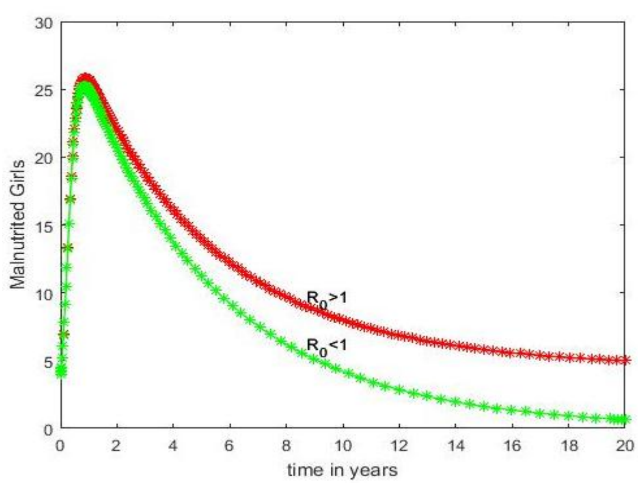

Fig. 5. Changes in malnourished girls group with R_0

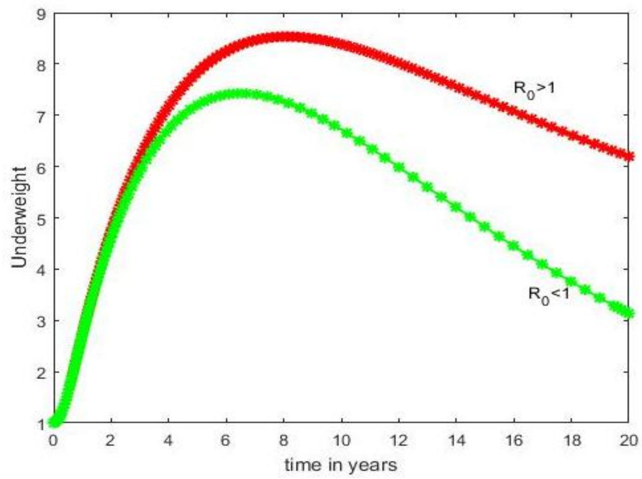

Fig. 6. Changes in underweight group with R_0

6. CONCLUSION

Here, in this paper the dynamics of underweight individual in the society because of malnutrition is studied. It has been established in this paper that healthy nutrition will reduce the new recruitments in the malnourished and underweight classes using control. It is desirable to have a healthy society. A healthy society equilibrium point is found which decides the malnutrition and underweight individuals. Also, the local and global stability at an equilibrium point is found which decides stability of the system. The control measures evaluated and optimized in section-4 with the intervention programmes like healthy life campaign and treatment shows that it has truly affected and has reduced the number of malnourished and underweight people. Results for different compartments have been calculated numerically which interprets that a constant treatment to the affected individuals in the initial stage will recover them faster but will take sufficient time to cure completely. At last, to live a better and healthy life, one must consume healthy and nutritive food. After all, “Health is Wealth”. Future research can be in terms of deciding (optimum) dosage of nutrients at the early stage. Also, one can incorporate different layers of the society to have more realistic analysis.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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