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Entropic Uncertainty Relations, Entanglement and Quantum Gravity Effects via the Generalised Uncertainty Principle

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Authors' contributions

This work was carried out in collaboration between both authors. Author GB defined the problem and proposed the initial approach for the solution. Author OG provided the right approach to the solution. The calculations and literature search were equally done by both authors. Author OG wrote the section on "Entropic Uncertainty Relations" while author GB wrote the rest of the paper. Both authors read and approved the final manuscript.

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ABSTRACT

We apply the generalised uncertainty principle (GUP) to the entropic uncertainty relation conditions on quantum entanglement. In particular, we study the GUP corrections to the Shannon entropic uncertainty condition for entanglement. We combine previous work on the Shannon entropy entanglement criterion for bipartite systems and the GUP corrections to the Shannon entropy for a single system to calculate the GUP correction for an entangled bipartite system. As in an earlier paper of the second author, which dealt with variance relations, it is shown that there is an increase in the upper bound for the entanglement condition upon the application of the generalised uncertainty principle. Necessary fundamental concepts of the generalised uncertainty principle, entanglement and the entropic uncertainty relations are also discussed. This paper puts together

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the concepts of entanglement, entropic uncertainty relations and the generalised uncertainty principle all of which have been separately discussed in pedagogical papers by Schroeder, Majernik et al., Blado et al. and Sprenger.

Keywords: Generalised uncertainty principle; entanglement; entropic uncertainty conditions; minimal length.

1. INTRODUCTION

The idea of a minimal length scale which limits the distance at which one can probe nature has been motivated by thought experiments involving gravitational effects in quantum mechanics and various attempts to formulate a quantum theory of gravity (such as string theory, loop quantum gravity, non-commutative geometry, etc.). In particular, string theory and certain thought experiments have been shown to give rise to a generalised uncertainty principle (GUP) from which models are developed which result in a minimal length scale [1]. In this paper, we will use the most widely used model which is through the modified commutation relations.

The Heisenberg Uncertainty Principle (HUP) of ordinary quantum mechanics is given by (variance form)

$$\Delta x \Delta p \geq \frac{1}{2}\ (\hbar = 1). \tag{1}$$

This is derivable from the commutator of *x* and *p*, $[x, p] = i$, using $(\Delta A)^2(\Delta B)^2 \geq \left(\frac{1}{2i}\langle[\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}]\rangle\right)^2$ from Griffiths [2], with *A* = *x* and *B* = *p*. String scattering has been shown to give rise to a modification of the HUP to a GUP [1, 3-5]. In this paper, we will use

$$\Delta x \Delta k \geq \frac{1}{2}(1 + \beta(\Delta k)^2) \tag{2}$$

as the form of the GUP which results from the modified commutation relation $[x, k] = i(1 + \beta k^2)$. *k* here is the momentum operator representation in terms of a GUP and β is the small positive GUP correction parameter. It is related to the HUP momentum operator by Pedram [6]

$$k = \frac{tan(\sqrt{\beta}p)}{\sqrt{\beta}}. \tag{3}$$

Eq. (2) exhibits a non-zero minimal length [7]. There is a rich literature [1,7-10] which discuss the GUP. GUP effects have been studied in quantum mechanical systems like the infinite square well, finite square well, simple harmonic oscillator, WKB tunneling effects as applied to alpha decay, quantum cosmogenesis and gravitational tunneling radiation. Implications in statistical physics and thermodynamics have also been studied. Most recently, investigations on GUP effects on the speed of gravitons, on the thermodynamics of the Schwarzchild-Tangherlini black hole and the entropic force law have also been advanced. Our short introduction above is meant to establish notation and discuss the most relevant equations for the discussions below.

Similar to Blado et al. [11], we study in the present paper, entanglement in continuous variable systems but instead of the variance form in Eq. (1), we will make use of entanglement conditions using the Shannon entropic uncertainty relations. The paper is organised as follows. In section 2 we give an elementary discussion of entanglement in terms of a finite dimensional bipartite system of qubits and briefly state how entanglement and the HUP can be related for a continuous variable system. Section 3 involves an account of the entropic uncertainty relations (Shannon entropy in particular). We work out in detail the Shannon entropic quantum entanglement criterion for a bipartite system involving the Einstein-Podolsky-Rosen-type (EPR-type) operators in section 4 to make the derivation of the GUP correction apparent in section 5. Some conclusions are given in section 6.

2. ENTANGLEMENT AND THE HUP

Particles are said to be entangled if their quantum states are somehow linked such that knowing the state of one particle determines the states of the others. Let us give an example using a qubit bipartite system to make this more explicit[1].

Suppose we have a system of two electrons (1 and 2). Let the spin state of electron 1 be

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \tag{4}$$

[1] *We will follow the discussion in [12].*

and electron 2 be

$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle \quad (5)$$

with $|0\rangle$ describes a particle with spin up and $|1\rangle$ a particle with spin down. The composite (pure) state of the system can be written as a tensor product (indicated by the operator $\otimes$), $|\psi\rangle = |\psi_1\rangle\otimes|\psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)\otimes(\alpha_2|0\rangle + \beta_2|1\rangle) = \alpha_1\alpha_2|0\rangle\otimes|0\rangle + \alpha_1\beta_2|0\rangle\otimes|1\rangle + \alpha_2\beta_1|1\rangle\otimes|0\rangle + \beta_1\beta_2|1\rangle\otimes|1\rangle$. A convenient notation is to let $|A\rangle\otimes|B\rangle = |AB\rangle$. So, we can write the preceding equation as

$$|\psi\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle. \quad (6)$$

Note that the expansion coefficients are such that $|\alpha_1\alpha_2|^2 + |\alpha_1\beta_2|^2 + |\alpha_2\beta_1|^2 + |\beta_1\beta_2|^2 = 1$.

Next, consider a state which can be formed from Eq. (6),

$$|\psi\rangle = \frac{1}{2\sqrt{2}}|00\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle \quad (7)$$

which is known to be the composite system made up of 2 states $|\psi_1\rangle$ and $|\psi_2\rangle$ of Eq. (4) and Eq. (5). We can factorise this into 2 states,

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right) = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\otimes\left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right) = |\psi_1\rangle\otimes|\psi_2\rangle \quad (8)$$

with $|\psi_1\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$ and $|\psi_2\rangle = \left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)$. Factorisable states are separable states [13]. Hence, we say that $|\psi\rangle$ in Eq. (7) is a separable state.

However, given a composite state, it is not always possible to factorise it into individual states! For example, consider the state,

$$|\psi'\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle. \quad (9)$$

If this is to factorise, then the right-hand side of Eq. (9) must be equal to the tensor product of two states. Let these two general states be as in Eq. (4) and Eq. (5). From Eq. (9) and Eq. (6), $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle$. Comparing coefficients, we get $\alpha_1\alpha_2 = \frac{1}{\sqrt{2}}$ so neither α_1 nor α_2 can be zero. We also get $\beta_1\beta_2 = \frac{1}{\sqrt{2}}$ so neither β_1 nor β_2 can be zero. However, we also get $\alpha_1\beta_2 = 0$ which means that either $\alpha_1 = 0$ or $\beta_2 = 0$. We have a contradiction. Hence, there is no solution and we cannot write $|\psi\rangle$ of Eq. (9) as a tensor product of the two states in Eq. (4) and Eq. (5). $|\psi'\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is an example of an entangled state. We can say that for the entangled state

$$|\psi'\rangle \neq |\psi_1\rangle\otimes|\psi_2\rangle. \quad (10)$$

As mentioned in the discussion above we can consider the state $|0\rangle$ to describe a particle with spin up and $|1\rangle$ a particle with spin down. So, for composite systems, the state $|10\rangle$ will describe the composite state in which particle 1 is spin down and particle 2 is spin up, etc. If we measure the spin of particle 1 in Eq. (9) to be up then we are sure that the second particle will have spin up also because of the composite state $|00\rangle$. This, however, is not true for Eq. (7). Due to the presence of the two terms $|00\rangle$ and $|01\rangle$, if particle 1 is spin up $|0\rangle$, there is a probability that the second particle is also spin down $|1\rangle$. Hence, Eq. (9) is called an entangled state because if we know the state of one particle, we surely know the state of the other unlike in Eq. (7).

The above example which makes use of a finite dimensional system, is a standard starting point in discussing the concept of entanglement. To explore the connection of the HUP and entanglement, we need to look at continuous variable systems which will involve the position and momentum operators. An excellent introductory discussion on entanglement beyond spin systems can be found in [14]. A more advanced treatment is given in [15].

Similar to Eq. (10)[2] an entangled composite state represented by $\Psi(x_1, x_2)$ in coordinate space or $\wp(p_1, p_2)$ in momentum space is such that $\Psi(x_1, x_2) \neq \boldsymbol{\psi_1}(x_1)\boldsymbol{\psi_2}(x_2)$ or $\wp(p_1, p_2) \neq \varphi_1(p_1)\varphi_2(p_2)$ where $\boldsymbol{\psi_i}(x_i)$ $(\varphi_i(p_i))$ could represent the coordinate (momentum) space wave function of particle *i*. With the states dependent on the position and momentum (which are continuous variables), and with the uncertainty principle expressed in terms of the position and momentum operators in Eq. (1), one could draw connections between the

[2] *We drop the tensor product notation for the particular representation of the state vectors in the continuous coordinate and momentum spaces.*

entanglement and the HUP [16-19] and subsequently with the GUP [11].

3. ENTROPIC UNCERTAINTY RELATIONS

For two observables that do not commute, such as position and momentum, Eq. (1) addresses a fundamental limit to the extent with which one can know about one of the particle's properties if the other has already been measured. For years, the favoured approach towards uncertainty in quantum mechanics was the use of standard deviations in Eq. (1) primarily because in probability and statistics, the dispersion is the most well-known gauge of uncertainty in measurements [20].

However, as pointed out by [20], it was not the ideal measure, indicated by two major flaws that appeared in the Robertson relation [21], the expression from which Eq. (1) can be derived when some non-commuting observables A and B have been specified to be position and momentum,

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle\Psi|\left[\hat{A},\hat{B}\right]|\Psi\rangle\right|. \quad (11)$$

The first weakness is that in a finite, N-dimensional Hilbert space, the lower bound in the right-hand side becomes dependent upon the state of the particle, which becomes especially problematic when the state is an eigenstate that gives an eigenvalue of the observable. Eq. (11) yields zero which basically eliminates the restriction on the variances of *A* and *B.* To see this, suppose $|\Psi\rangle$ is an eigenstate of the observable $\hat{B}$ such that $\hat{B}|\Psi\rangle = b|\Psi\rangle$ with the eigenvalue *b* as real. Clearly, $\langle\Psi|\hat{A}\hat{B}\big|\Psi\rangle = \langle\Psi|\hat{B}\hat{A}\big|\Psi\rangle = b\langle\Psi|\hat{A}|\Psi\rangle$ which causes the vanishing commutator of Eq. (11).

The second weakness is that for conjugate variables with continuous probability distributions, the dispersion may not be the useful measure of uncertainty especially if the distributions have several peaks. This is especially true for applications in information theory where standard deviation does not play an important role [22].

To overcome these issues, alternative uncertainty relations were sought out by physicists. Recently, an approach utilising entropic uncertainty relations has been gaining traction due to the relations' clear validity as mathematical inequalities [23]. Consequently, it has been postulated that such relations may bear physical significance in measurements and so the search of a useful relation, if any exist, for analysis of quantum mechanical systems has been the topic of debate [24,25]. The desirable features of the entropic uncertainty relations over the HUP have been discussed in [22].

In favour of the view that entropic uncertainty relations can be useful in experiments, many physicists attempt to explain what entropy can mean in a quantum context. In one particular definition, entropy can be seen as the amount of information one lacks when one witnesses an event that could have many outcomes [26]. This information can be used to characterise its source based not on what the information is about, but on how much of it is transmitted compared to how much of it can be transmitted.

To understand the definition of the particular entropic function we will be using, let us first go back to the Boltzmann entropy $S = k_B \ln W$, of classical thermodynamics and statistical mechanics [27,28] where k_B is the Boltzmann's constant and *W* is the number of microscopic configurations that a system can be in, assuming equiprobable configurations. Hence $W = 1/P$ where *P* is the probability of each configuration. Thus $S/k_B = -\ln(P)$. However, if the different configurations are not equiprobable, then we need to add the weight-factor of them occurring as we sum over the different probabilities P_i,

$$S/k_B = -\sum_i P_i \, ln(P_i)\,. \quad (12)$$

From Eq. (12), Shannon [26] defined the (dimensionless) average information entropy content of a random variable X (called the Shannon entropy) as

$$H \equiv \sum_x P_X(x)\, ln(1/P_X(x)) = -\sum_x P_X(x)\, ln\big(P_X(x)\big), \quad (13)$$

where $P_X(x)$ is the probability distribution of the random variable X to have the measured value x. Note that if the probability of measuring X as x is 1 then there is no information entropy or no uncertainty. The lower the probability of measuring x, the higher the information entropy or uncertainty. Hence information entropy can be considered a measure of uncertainty.

Extending Eq. (13) to continuous variables, we get the differential Shannon entropy (or we will refer to this as simply "Shannon entropy" in the succeeding sections) as

$$H = -\int_{x_1}^{x_2} P(x)\, ln\big(P(x)\big)\, dx \tag{14}$$

where $P(x)$ is the probability distribution of the continuous variable *x*.

It is apparent that we can define the Shannon entropy involving position and momentum probability distributions from Eq. (14) as

$$H[w] = -\int_{-\infty}^{\infty} dx\, w(x)\, ln\, w(x);$$
$$H[v] = -\int_{-\infty}^{\infty} dp\, v(p)\, ln\, v(p) \tag{15}$$

where $w(x) \equiv |\psi(x)|^2$ and $v(p) \equiv |\varphi(p)|^2$. Bialynicki-Birula and Mycielski [29] derived an entropic uncertainty relation ("BBM inequality") involving the Shannon entropy similar to the HUP

$$H[w] + H[v] \geq ln(\pi e). \tag{16}$$

The above expression owes its validity to a fact that Heisenberg himself acknowledged—to gain information about the position of a particle, one must lose information about the momentum of that same particle and vice versa due to their nature as conjugate variables [30]. Essentially, as the entropy of one property decreases, the entropy of the other must increase to maintain the total uncertainty of the particle above the limit shown in the right-hand side of Eq. (16). In addition, although the HUP describes uncertainty due to a lack of the simultaneous existence of precise values for both position and momentum, it ignores the uncertainty induced by the measurement of an actual instrument, which Eq. (16) can account for [23]. Coles, et al. [22] discussed how Eq. (16) overcomes the two weaknesses of the HUP discussed above. In particular, one can note that since the right hand side of Eq. (16) is a constant, it is not dependent on the state [31] and will not have the problem that the HUP in Eq. (11) has in which the lower bound vanishes for the eigenstates of the corresponding physical observable operators. The left hand side of Eq. (16) has been calculated for some systems. Their values can be equal to $ln(\pi e)$ for the ground state of the simple harmonic oscillator [32] and is shown to increase for higher states. It has also been calculated for the infinite square well and starts off slightly greater than $ln(\pi e)$ and then increases for higher states [20,33].

Since the BBM inequality is a reliable representation of the uncertainty inherent in knowing the properties of one particle, it will also be useful when considering two particles which apparently involves more information. This forms the basis of an entropic uncertainty relation that can be used to detect a specific characteristic of the two-particle system, namely entanglement, due to the fact that the measurement of one correlated particle gives rise to information on the other particle as well, resulting to information about both particles instead of only that which was measured. This mechanism decreases the total uncertainty of the system beyond its expected limit. Identification of entangled particles has always proven inconvenient with factorisation since the process can sometimes be tedious and it is difficult to be certain that some composite states are definitively inseparable. A lack of factorisability can indicate that particles are entangled, but it can also mean that the composite state is simply very complex, so an entropic criterion that can guarantee entanglement would be very useful indeed.

4. SHANNON ENTROPY QUANTUM ENTANGLEMENT CRITERION

The following discussion will follow closely Walborn, et al. [31]. Considering a bipartite system, we define operators similar to the annihilation operators defined in [34], which are linear combinations of the position and momentum operators x_j and p_j with j =1, 2, pertaining to subsystems 1 and 2, and the usual commutation relation, $[x_j, p_k] = i\delta_{jk}$

$$r_j = cos(\theta_j)x_j + sin(\theta_j)\, p_j\ ,$$
$$s_j = cos(\theta_j)p_j - sin(\theta_j)\, x_j\ . \tag{17}$$

From Eq. (17), we define further the operators, $r_\pm = r_1 \pm r_2$ and $s_\pm = s_1 \pm s_2$. As in [35], we study EPR-type operators by letting $\theta_1 = \theta_2 = 0$ in Eq. (17) which give, similar to the general $r_\pm$ and $s_\pm$ operators,

$$x_\pm = x_1 \pm x_2\ ;\ p_\pm = p_1 \pm p_2\ . \tag{18}$$

Consider a separable pure state $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ with the corresponding wavefunction in coordinate space, $\Psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$ and in momentum space $\wp(p_1, p_2) = \varphi_1(p_1)\varphi_2(p_2)$. A change in variables in Eq. (18), yields

$$\frac{1}{\sqrt{2}}\psi_1\left(\frac{x_+ + x_-}{2}\right)\psi_2\left(\frac{x_+ - x_-}{2}\right) = \Psi(x_+, x_-)\ ;$$
$$\frac{1}{\sqrt{2}}\varphi_1\left(\frac{p_+ + p_-}{2}\right)\varphi_2\left(\frac{p_+ - p_-}{2}\right) = \wp(p_+, p_-). \tag{19}$$

Let us compute the Shannon entropy associated with the measurement of $x_\pm$ and $p_\pm$. From the definition of the Shannon entropy in Eq. (15),

$H[w_\pm] = -\int_{-\infty}^{\infty} dx_\pm w_\pm(x_\pm)\, ln\left(w_\pm(x_\pm)\right)$; $H[v_\pm] = -\int_{-\infty}^{\infty} dp_\pm v_\pm(p_\pm)\, ln\left(v_\pm(p_\pm)\right)$, where

$w_\pm(x_\pm) = w_\pm(x_1, x_2)$ and $v_\pm(p_\pm) = v_\pm(p_1, p_2)$ are the probability distributions of $x_\pm$ and $p_\pm$ respectively. From Eq. (19), these are given by

$$w_\pm(x_\pm) = \int_{-\infty}^{\infty} dx_\mp |\Psi(x_+, x_-)|^2 = \frac{1}{2}\int_{-\infty}^{\infty} dx_\mp |\psi_1|^2|\psi_2|^2\ ;$$
$$v_\pm(p_\pm) = \int_{-\infty}^{\infty} dp_\mp |\wp(p_+, p_-)|^2 = \frac{1}{2}\int_{-\infty}^{\infty} dp_\mp |\varphi_1|^2|\varphi_2|^2. \qquad (20)$$

Let us define $w_i(x_i) \equiv |\psi_i(x_i)|^2$ and $v_i(p_i) \equiv |\varphi_i(p_i)|^2$. Hence Eq. (20) becomes

$$w_\pm(x_\pm) = \frac{1}{2}\int_{-\infty}^{\infty} dx_\mp w_1\left(\frac{x_+ + x_-}{2}\right) w_2\left(\frac{x_+ - x_-}{2}\right) \text{ and}$$
$$v_\pm(p_\pm) = \frac{1}{2}\int_{-\infty}^{\infty} dp_\mp v_1\left(\frac{p_+ + p_-}{2}\right) v_2\left(\frac{p_+ - p_-}{2}\right) \qquad (21)$$

For convenience, as will be apparent later, we make a further change in variables in terms of $\frac{x_+ + x_-}{2} = x_1 = x$ and so $\frac{x_+ - x_-}{2} = \mp x \pm x_\pm$ and similarly $\frac{p_+ + p_-}{2} = p_1 = p$ and so $\frac{p_+ - p_-}{2} = \mp p \pm p_\pm$. Eq. (21) becomes $w_\pm = \int_{-\infty}^{\infty} dx w_1(x) w_2(\mp x \pm x_\pm)$; $v_\pm = \int_{-\infty}^{\infty} dp v_1(p) v_2(\mp p \pm p_\pm)$ which in terms of the convolution operation "*" can be written simply as

$$w_\pm = w_1 * w_2^\pm\ ,\quad v_\pm = v_1 * v_2^\pm \text{ with } w_2^\pm = w_2(\pm x) \text{ and } v_2^\pm = v_2(\pm p). \qquad (22)$$

From the inequality [26, 36],

$$e^{2H[A*B]} \geq e^{2H[A]} + e^{2H[B]} \qquad (23)$$

and with [26]

$$H[w_2^+] = H[w_2^-] \text{ and } H[v_2^+] = H[v_2^-] \qquad (24)$$

we get from Eq. (22)

$$H[w_\pm] = H\left[w_1 * w_2^\pm\right] \geq \frac{1}{2} ln\left\{e^{2H[w_1]} + e^{2H[w_2]}\right\};$$
$$H[v_\mp] = H\left[v_1 * v_2^\mp\right] \geq \frac{1}{2} ln\left\{e^{2H[v_1]} + e^{2H[v_2]}\right\}. \qquad (25)$$

The preceding equation leads to state-dependent inequalities satisfied by separable pure states.

$$H[w_\pm] + H[v_\mp] \geq \frac{1}{2} ln\left\{\left(e^{2H[w_1]} + e^{2H[w_2]}\right)\left(e^{2H[v_1]} + e^{2H[v_2]}\right)\right\} \equiv \mathrm{h}\left[w_i, v_j\right] \qquad (26)$$

Hence if

$$H[w_\pm] + H[v_\mp] < \frac{1}{2} ln\left\{\left(e^{2H[w_1]} + e^{2H[w_2]}\right)\left(e^{2H[v_1]} + e^{2H[v_2]}\right)\right\} \equiv \mathrm{h}\left[w_i, v_j\right] \qquad (27)$$

then we have entangled states.

From the BBM inequality [29], we can write

$$H\left[w_j\right] + H\left[v_j\right] \geq ln(\pi e) \text{ with } j = 1, 2. \qquad (28)$$

This yields from Eq. (26),

$$H[w_\pm] + H[v_\mp] \geq \frac{1}{2} ln\left\{2(\pi e)^2 + e^{2H[w_1] + 2H[v_2]} + e^{2H[w_2] + 2H[v_1]}\right\}. \qquad (29)$$

Using Eq. (28) once again, we can write Eq. (29) in terms of only w_j (or v_j) as $H[w_\pm] + H[v_\mp] \geq \frac{1}{2} ln\left\{2(\pi e)^2 + (\pi e)^2\left[e^{2(H[v_2] - H[v_1])} + e^{-2(H[v_2] - H[v_1])}\right]\right\}$ or simply

$$H[w_\pm] + H[v_\mp] \geq \frac{1}{2} ln\{2(\pi e)^2(1 + cosh\Delta H)\} \geq ln(2\pi e) \qquad (30)$$

where $\Delta H \equiv H[v_2] - H[v_1]$. One can compute that the minimum value of the argument of the natural logarithm function occurs when $\Delta H = 0$ which then yields the minimum value $4(\pi e)^2 = (2\pi e)^2$. Substituting this value, gives the last inequality in the preceding equation. Although Eq. (30) is a weaker set of state-independent inequalities, states with

$$H[w_\pm] + H[v_\mp] < ln(2\pi e) \qquad (31)$$

are still entangled states. However entangled states can still exist for $ln(2\pi e) \leq H[w_\pm] + H[v_\mp] < h\left[w_i, v_j\right]$ owing to the stronger inequalities in Eq. (26) and Eq. (27).

Let us next consider mixed states. Mixed states are more commonly encountered experimentally since a system can be in any of a number of states in varying probabilities which can occur when we prepare a similar system several times. Note that a mixed state is different from a state formed by a superposition of states (which can be a pure state). A bipartite separable mixed state is more conveniently described in terms of the density matrix ρ.

$$\rho = \sum_m \lambda_m \rho_{1m} \otimes \rho_{2m}\ , \quad \text{where} \quad \rho_{im} = |\psi_{im}\rangle\langle\psi_{im}|\ , \quad i = 1,2,\ \lambda_m \geq 0,\ \sum_m \lambda_m = 1. \tag{32}$$

Note that $\psi_{1m} = \left(\psi_1(r_1)\right)_m$ is the *m*th pure state for subsystem 1 and $\psi_{2m} = \left(\psi_2(r_2)\right)_m$ is the *m*th pure state for subsystem 2. We can write the probability distribution for $x_\pm$ and $p_\pm$ as

$$w_\pm = \sum_m \lambda_m w_{m\pm},\ v_\pm = \sum_m \lambda_m v_{m\pm} \tag{33}$$

where $w_{m\pm}$ ($v_{m\pm}$) is the probability distribution to detect $x_\pm$ ($p_\pm$) for each pure state ψ_{im}.

Since the Shannon entropy is a concave function [36], we have

$$H[w_\pm] = H[\sum_m \lambda_m w_{m\pm}] \geq \sum_m \lambda_m H[w_{m\pm}]\ ; \quad H[v_\mp] = H[\sum_m \lambda_m v_{m\mp}] \geq \sum_m \lambda_m H[v_{m\mp}]. \tag{34}$$

Similarly, from Eq. (25), we get

$$H[w_\pm] \geq H[\sum_m \lambda_m w_{m\pm}] \geq \sum_m \lambda_m \tfrac{1}{2} ln\{e^{2H[w_{1m}]} + e^{2H[w_{2m}]}\};\quad H[v_\mp] \geq H[\sum_m \lambda_m v_{m\mp}] \geq \sum_m \lambda_m \tfrac{1}{2} ln\{e^{2H[v_{1m}]} + e^{2H[v_{2m}]}\}. \tag{35}$$

Hence, adding the preceding equations,

$$H[w_\pm] + H[v_\mp] \geq \sum_m \lambda_m \tfrac{1}{2} ln\left[\left(e^{2H[w_{1m}]} + e^{2H[w_{2m}]}\right)\left(e^{2H[v_{1m}]} + e^{2H[v_{2m}]}\right)\right] \equiv \hbar[w_{im}, v_{jm}] \tag{36}$$

for separable states where $i, j = 1,2$. Hence for entangled mixed states we have,

$$H[w_\pm] + H[v_\mp] < \hbar[w_{im}, v_{jm}]. \tag{37}$$

Going back to Eq. (34), we can write $H[w_\pm] + H[v_\mp] \geq \sum_m \lambda_m \left(H[w_{m\pm}] + H[v_{m\mp}]\right)$ and from Eq. (30), we get $H[w_\pm] + H[v_\mp] \geq \sum_m \lambda_m \left(ln(2\pi e)\right) = \left(ln(2\pi e)\right) \sum_m \lambda_m = \left(ln(2\pi e)\right) \cdot 1$ from Eq. (32). Hence, $H[w_\pm] + H[v_\mp] \geq ln(2\pi e)$ even for separable mixed states and similarly we get the same weaker conditions for entanglement in Eq. (31) for mixed states

$$H[w_\pm] + H[v_\mp] < ln(2\pi e). \tag{38}$$

5. GUP-CORRECTION TO THE SHANNON ENTROPY QUANTUM ENTANGLEMENT CRITERION

Having set the notation and discussed the Shannon Entropy Quantum Entanglement Criterion in section 4, we use the results of [37, 6] to get the GUP correction. As discussed in section 1, the GUP-modified commutation relation is now given by $[x_i, k_j] = i\delta_{ij}(1 + \beta k_i^2)$ with *i*, and *j* as in section 0 above. From Eq. (3), p_i and k_i are related by

$$k_i = \frac{1}{\sqrt{\beta}} tan(\sqrt{\beta} p_i). \tag{39}$$

We now have for the *k* momentum operator, analogous to $\wp(p_+, p_-)$,

$$\Phi(k_+, k_-) = \frac{1}{\sqrt{2}} \phi_1\left(\frac{k_+ + k_-}{2}\right) \phi_2\left(\frac{k_+ - k_-}{2}\right) \tag{40}$$

with

$$k_\pm = k_1 \pm k_2 \text{ and } \phi_1 = \phi_1(k_1),\ \phi_2 = \phi_2(k_2) \tag{41}$$

and the corresponding Shannon entropy $H[u_\pm] = -\int_{-\infty}^{\infty} dk_\pm u_\pm(k_\pm)\, ln\left(u_\pm(k_\pm)\right)$ where $u_\pm(k_\pm) = u_\pm(k_1, k_2)$ is the probability distribution of $k_\pm$. Hence from Eq. (40),

$$u_\pm(k_\pm) = \int_{-\infty}^{\infty} dk_\mp |\Phi(k_+, k_-)|^2 = \frac{1}{2}\int_{-\infty}^{\infty} dk_\mp |\phi_1|^2 |\phi_2|^2\ . \tag{42}$$

Defining $u_i(k_i) \equiv |\phi_i(k_i)|^2$, Eq. (42) becomes

$$u_\pm(k_\pm) = \frac{1}{2}\int_{-\infty}^{\infty} dk_\mp u_1\left(\frac{k_+ + k_-}{2}\right) u_2\left(\frac{k_+ - k_-}{2}\right). \tag{43}$$

From Eq. (41), we define $\frac{k_+ + k_-}{2} = k_1 \equiv k$, and so we get $\frac{k_+ - k_-}{2} = \mp k \pm k_\pm$ then Eq. (43) becomes $u_\pm(k_\pm) = \int_{-\infty}^{\infty} dk u_1(k) u_2(\mp k \pm k_\pm)$ which in terms of the convolution operation " $*$ " becomes $u_\pm(k_\pm) = u_1 * u_2^\pm$ with $u_2^\pm \equiv u_2(\pm k)$. Using Eq. (23) and Eq. (24) (with *v* replaced by *u*), we get similar to the second part of Eq. (25),

$$H[u_\mp] = H\left[u_1 * u_2^\mp\right] \geq \frac{1}{2} ln\left\{e^{2H[u_1]} + e^{2H[u_2]}\right\}. \tag{44}$$

The probability distributions of the operators *k* and *p* are such that $u_i(k_i)dk_i = v_i(p_i)dp_i$ or $u_i(k_i) = v_i(p_i)\frac{dp_i}{dk_i}$ with $i = 1,2$. Using Eq. (39), we get

$$u_i(k_i) = \frac{v_i(p_i)}{1 + \beta k_i^2}\ . \tag{45}$$

From the definition of the Shannon entropy Eq. (15) and Eq. (45), $H[u_i] = -\int_{-\infty}^{\infty} u_i(k_i)\ ln\, u_i(k_i)\ dk_i =$

$-\int_{-\infty}^{\infty}\frac{v_i(p_i)}{1+\beta k_i^2} ln\left(\frac{v_i(p_i)}{1+\beta k_i^2}\right) dk_i =$
$-\int_{-\infty}^{\infty}\frac{v_i(p_i)}{1+\beta k_i^2} ln\, v_i(p_i)\, dk_i + \int_{-\infty}^{\infty}\frac{v_i(p_i)}{1+\beta k_i^2} ln\left(1+\beta k_i^2\right) dk_i =$
$-\int_{-\infty}^{\infty}\frac{v_i(p_i)}{1+\beta k_i^2} ln\, v_i(p_i)\, dk_i + \int_{-\infty}^{\infty} u_i(k_i)\, ln\left(1+\beta k_i^2\right) dk_i$.
Finally, using again Eq. (39) for a change in the integrating variable in the first integral yields $H[u_i] = -\int_{-p_0}^{p_0} v_i(p_i)\, ln\left(v_i(p_i)\right) dp_i + \int_{-\infty}^{\infty} u_i(k_i)\, ln(1+\beta k_i^2)\, dk_i$ with $p_0 \equiv \frac{\pi}{2\sqrt{\beta}}$. Hence, we can write

$H[u_i] = H[v_i] + \eta_i(\beta)$ with
$$\eta_i(\beta) \equiv \int_{-\infty}^{\infty} u_i(k_i)\, ln(1+\beta k_i^2)\, dk_i > 0\ . \quad (46)$$

$\eta_i(\beta)$ basically gives the GUP correction. From Eq. (46) and Eq. (44), we get

$$H[u_{\mp}] \geq \frac{1}{2} ln\left\{e^{2H[v_1]+2\eta_1(\beta)} + e^{2H[v_2]+2\eta_2(\beta)}\right\}. \quad (47)$$

Combining the preceding equation with the inequality for $w_{\pm}$ in Eq. (25), we get $H[w_{\pm}] + H[u_{\mp}] \geq \frac{1}{2} ln\left\{e^{2\eta_1(\beta)}e^{2H[v_1]} + e^{2\eta_2(\beta)}e^{2H[v_2]}\right\} + \frac{1}{2} ln\left\{e^{2H[w_1]} + e^{2H[w_2]}\right\}$ or

$$H[w_{\pm}] + H[u_{\mp}] \geq \frac{1}{2} ln\left\{\left(e^{2H[w_1]} + e^{2H[w_2]}\right)\left(e^{2\eta_1(\beta)}e^{2H[v_1]} + e^{2\eta_2(\beta)}e^{2H[v_2]}\right)\right\} \equiv h_{GUP}\left(w_i, v_j\right) \quad (48)$$

for separable pure states. Since from Eq. (46) $\eta_i(\beta = 0) = 0$ for the non-GUP case, Eq. (48) reduces to Eq. (26) as expected. Hence, if $H[w_{\pm}] + H[u_{\mp}] < h_{GUP}\left(w_i, v_j\right)$ then we have entangled states. Clearly since $\eta_i(\beta) > 0$ in Eq. (46), we see from Eq. (26) and Eq. (48) that $h_{GUP}\left(w_i, v_j\right) > h\left[w_i, v_j\right]$. Hence with Eq. (27) and Eq. (48) we have a higher upper bound for the GUP-corrected entanglement condition. It is interesting to note that the calculation of the GUP effects on the entanglement condition using the variance form of the uncertainty relations also resulted in an increase in the upper bound of the corresponding entanglement conditions [11].

We now derive the corresponding GUP-corrected expression for Eq. (30). Since the BBM inequality has been shown to still hold in the GUP framework [6], we can combine Eq. (46) and Eq. (28) to yield

$$H[u_i] + H[w_i] \geq ln(\pi e) + \eta_i(\beta) \quad (49)$$

which yields $e^{H[u_i]+H[w_i]} \geq (\pi e)e^{\eta_i(\beta)}$ or
$$e^{2(H[u_i]+H[w_i])} \geq (\pi e)^2 e^{2\eta_i(\beta)}. \quad (50)$$

From Eq. (44) and Eq. (25), we get $H[w_{\pm}] + H[u_{\mp}] \geq \frac{1}{2} ln\left(e^{2H[w_1]} + e^{2H[w_2]}\right) + \frac{1}{2} ln\left\{e^{2H[u_1]} + e^{2H[u_2]}\right\} = \frac{1}{2} ln\left\{\left(e^{2H[w_1]} + e^{2H[w_2]}\right)\left(e^{2H[u_1]} + e^{2H[u_2]}\right)\right\}$ or
$$H[w_{\pm}] + H[u_{\mp}] \geq \frac{1}{2} ln\left(e^{2H[w_1]+2H[u_1]} + e^{2H[w_2]+2H[u_2]} + e^{2H[w_1]+2H[u_2]} + e^{2H[w_2]+2H[u_1]}\right). \quad (51)$$

Because of Eq. (50), we can replace the first two terms of Eq. (51) to yield

$$H[w_{\pm}] + H[u_{\mp}] \geq \frac{1}{2} ln\left\{(\pi e)^2 e^{2\eta_1(\beta)} + (\pi e)^2 e^{2\eta_2(\beta)} + e^{2H[w_1]+2H[u_2]} + e^{2H[w_2]+2H[u_1]}\right\}. \quad (52)$$

Let us look at the third term of the preceding equation. Using Eq. (49) to eliminate $H[w_1]$, we have $e^{2H[w_1]+2H[u_2]} \geq e^{2(ln(\pi e)+\eta_1(\beta)-H[u_1])+2H[u_2]} = (\pi e)^2 e^{2\eta_1(\beta)} e^{2(H[u_2]-H[u_1])}$. Similarly, we have $e^{2H[w_2]+2H[u_1]} \geq (\pi e)^2 e^{2\eta_2(\beta)} e^{-2(H[u_2]-H[u_1])}$. We can then replace the last two terms in Eq. (52) to give $H[w_{\pm}] + H[u_{\mp}] \geq \frac{1}{2} ln\left((\pi e)^2 e^{2\eta_1(\beta)} + (\pi e)^2 e^{2\eta_2(\beta)} + (\pi e)^2 e^{2\eta_1(\beta)} e^{2(H[u_2]-H[u_1])} + (\pi e)^2 e^{2\eta_2(\beta)} e^{-2(H[u_2]-H[u_1])}\right)$.
The right-hand side readily simplifies to

$$H[w_{\pm}] + H[u_{\mp}] \geq ln(\pi e) + \frac{1}{2} ln\left\{\left(e^{2\eta_1(\beta)} + e^{2\eta_2(\beta)}\right) + \left(e^{2\eta_1(\beta)} e^{2\Delta H_{GUP}} + e^{2\eta_2(\beta)} e^{-2\Delta H_{GUP}}\right)\right\} \quad (53)$$

with $\Delta H_{GUP} \equiv H[u_2] - H[u_1]$. As in section 4, Eq. (30), let us attempt to rewrite the inequality independent of the Shannon entropy difference ΔH_{GUP} . We can solve for the minimum of $e^{2\eta_1(\beta)} + e^{2\eta_2(\beta)} + e^{2\eta_1(\beta)} e^{2\Delta H_{GUP}} + e^{2\eta_2(\beta)} e^{-2\Delta H_{GUP}}$. It occurs at $\Delta H_{GUP} = \frac{1}{2} ln\left(\frac{e^{2\eta_2(\beta)}}{e^{2\eta_1(\beta)}}\right)$. This yields a minimum value of $e^{2\eta_1(\beta)} + e^{2\eta_2(\beta)} + 2e^{\eta_1(\beta)} e^{\eta_2(\beta)} = \left(e^{\eta_1(\beta)} + e^{\eta_2(\beta)}\right)^2$. We can substitute this into the argument of the natural logarithm function in Eq. (53) to get $H[w_{\pm}] + H[u_{\mp}] \geq ln(\pi e) + ln\left(e^{\eta_1(\beta)} + e^{\eta_2(\beta)}\right)$ or

$$H[w_{\pm}] + H[u_{\mp}] \geq ln(2\pi e) + ln\left(\frac{e^{\eta_1(\beta)} + e^{\eta_2(\beta)}}{2}\right) \quad (54)$$

Eq. (54) is the GUP-corrected version of Eq. (30). Unlike Eq. (30), the preceding equation is not state-independent. The GUP-corrected version of Eq. (31) for entangled states is then

$$H[w_{\pm}] + H[u_{\mp}] < ln(2\pi e) + ln\left(\frac{e^{\eta_1(\beta)} + e^{\eta_2(\beta)}}{2}\right). \quad (55)$$

As before the upper bound for entangled states is increased when compared to Eq. (31).

The GUP correction for mixed states expressed in Eq. (32) follows naturally. Similar to Eq. (33),

$$w_{\pm} = \sum_m \lambda_m w_{m\pm},\; u_{\pm} = \sum_m \lambda_m u_{m\pm} \tag{56}$$

where $u_{m\pm}$ is the probability distribution to detect $k_{\pm}$ for each pure state ψ_{im}.

Similar to Eq. (46),

$$H[u_{im}] = H[v_{im}] + \eta_{im}(\beta) \quad \text{with} \quad \eta_{im}(\beta) \equiv \int_{-\infty}^{\infty} u_{im}(k_i)\, ln(1+\beta k_i^2)\; dk_i > 0,\ \eta_{im}(\beta),\ i = 1, 2 \tag{57}$$

giving the GUP correction. Using the concavity of the Shannon entropy function, Eq. (56) gives

$$H[u_{\mp}] = H[\sum_m \lambda_m u_{m\mp}] \geq \sum_m \lambda_m H[u_{m\mp}]. \tag{58}$$

Similar to Eq. (47), $H[u_{m\mp}] \geq \frac{1}{2} ln\{e^{2H[v_{1m}]+2\eta_{1m}(\beta)} + e^{2H[v_{2m}]+2\eta_{2m}(\beta)}\}$. Hence Eq. (58) becomes $H[u_{\mp}] \geq \sum_m \lambda_m \frac{1}{2} ln\{e^{2H[v_{1m}]+2\eta_{1m}(\beta)} + e^{2H[v_{2m}]+2\eta_{2m}(\beta)}\}$. Adding the preceding equation to Eq. (35),

$$H[w_{\pm}] + H[u_{\mp}] \geq \sum_m \lambda_m \frac{1}{2} ln\{\left(e^{2H[w_{1m}]} + e^{2H[w_{2m}]}\right)\left(e^{2H[v_{1m}]+2\eta_{1m}(\beta)} + e^{2H[v_{2m}]+2\eta_{2m}(\beta)}\right)\} \equiv \hbar_{GUP}\left(w_{im}, v_{jm}\right) \tag{59}$$

for separable states. Hence for entangled mixed states, we have $H[w_{\pm}] + H[u_{\mp}] < \hbar_{GUP}\left(w_{im}, v_{jm}\right)$. Hence, similar to the conclusion from Eq. (48), we get a higher upper bound for the GUP-corrected entanglement condition as compared to Eq. (37) since $\hbar_{GUP}\left(w_{im}, v_{jm}\right) > h_{GUP}\left(w_{im}, v_{jm}\right)$ as in the pure state case.

Returning to Eq. (34) and Eq. (58), and similar to Eq. (54), $H[w_{\pm}] + H[u_{\mp}] \geq \sum_m \lambda_m (H[w_{m\pm}] + H[u_{m\mp}]) \geq \sum_m \lambda_m \left(ln(2\pi e) + ln\left[\frac{e^{\eta_{1m}(\beta)}+e^{\eta_{2m}(\beta)}}{2}\right]\right) = \sum_m \lambda_m (ln(2\pi e)) + \sum_m \lambda_m\, ln\left[\frac{e^{\eta_{1m}(\beta)}+e^{\eta_{2m}(\beta)}}{2}\right]$ and from Eq. (32), $H[w_{\pm}] + H[u_{\mp}] \geq ln(2\pi e) + \sum_m \lambda_m\, ln\left[\frac{e^{\eta_{1m}(\beta)}+e^{\eta_{2m}(\beta)}}{2}\right]$. The entanglement criterion is then

$$H[w_{\pm}] + H[u_{\mp}] < ln(2\pi e) + \sum_m \lambda_m\, ln\left[\frac{e^{\eta_{1m}(\beta)}+e^{\eta_{2m}(\beta)}}{2}\right]. \tag{60}$$

Eq. (60) is the GUP-corrected expression for Eq. (38) for mixed states (generalised for both pure states and mixed states) which shows an increase in the upper bound of Eq. (38). It is apparent that Eq. (60) reduces to Eq. (55) for a pure state with a weight-factor of 1.

6. CONCLUSIONS

The study of the effects of the minimal length and the generalised uncertainty principle on entanglement involving continuous variable systems has been confined to the variance form of the HUP [11]. Since the more modern approach of expressing the uncertainty principle is through the use of entropic measures [38], as discussed in section 0 above, we conducted a study of how the entanglement criterion using the Shannon entropic uncertainty relation is modified by the generalised uncertainty principle. As expected, the upper bounds of the entanglement criteria for pure and mixed states in bipartite systems are increased just like in the variance form [11] from which we can infer as in [11] that the increase in the bounds due to GUP effects enhances entanglement making the quantum effects more pronounced. As mentioned in section 1, the GUP is a result of formulating a quantum theory of gravity leading to the inclusion of gravitational effects in quantum mechanics. Intuitively, we expect the introduction of gravitational effects via the GUP, to increase the entropy and hence the uncertainty leading to a higher upper bound for entanglement.

It should also be noted that the main difference between the weaker inequality entanglement criteria using the Shannon entropy for pure and mixed states without the GUP (Eq. (31) and Eq. (38)) and with the GUP (Eq. (55) and Eq. (60)) is the state-dependence due to the GUP correction $\eta_i(\beta)$ and $\eta_{im}(\beta)$ defined in Eq. (46) and Eq. (57). However, a state-independent entanglement criterion can apparently be derived using the Rényi and Tsallis entropies since state-independent entropic uncertainty relations with GUP have been obtained in [37]. Since the entropic bounds are increased we expect also an increase in the entanglement criteria upper bounds.

We now comment on a possible future extension of our work. As mentioned above, with the GUP correction obtained for the Rényi and Tsallis entropic uncertainty relations in [37], it will be interesting to study carefully the GUP effect on the entanglement criteria based on the Rényi entropy as a generalisation of the Shannon entropy. Preliminary work on the GUP effects on the Rényi entropic uncertainty relations has indeed shown an increase in the entanglement criteria upper bound [Gadea O. Derivation of the Rényi Entropic Uncertainty Relation and Entanglement Criterion. 2018; unpublished].

As a final brief remark, we speculate on a link between entanglement and gravity. Entanglement is a purely quantum effect. By applying the GUP (which results from a quantum theory of gravity) to one aspect of determining entanglement by the entropic uncertainty relations, we see a relationship between gravity and entanglement. However, the connection has appeared elsewhere. It has been proposed [39] that the entanglement (measured in terms of entanglement von Neumann entropy) of degrees of freedom in quantum systems can give rise to the emergence of spacetime in the gravity picture from field theory with the gravity- gauge theory correspondence (the so called anti-deSitter space-conformal field theory or AdS-CFT correspondence) [40-42]. Whereas these studies indicate the effect of quantum entanglement on gravity, our work opens up an avenue of the effect of gravity on quantum entanglement. Gravity and quantum entanglement seem to be mutually related.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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