Global Stability of Almost Periodic Solution of a Discrete Multispecies Gilpin-Ayala Mutualism System

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Author’s contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This paper discusses a discrete multispecies Gilpin-Ayala mutualism system. We first study the permanence and global attractivity of the system. Assume that the coefficients in the system are almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive and uniformly asymptotically stable by constructing a suitable Liapunov function, respectively. Two examples together with numerical simulation indicate the feasibility of the main results.

Keywords: Almost periodic solution; discrete; gilpin-ayala mutualism system; permanence; global attractivity; uniformly asymptotically stable.

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1 Introduction

The mutualism system has been studied by more and more scholars. Topics such as permanence, global attractivity and global stability of continuous differential mutualism system were extensively investigated (see [2,3,4,5] and the references cited therein). Xia et al. [2] studied a Lotka-Volterra type mutualism system with several delays. Some new and interesting sufficient conditions are obtained for the global existence of positive periodic solutions of the mutualism system. Their method is based on Mawhin’s coincidence degree and novel estimation techniques for the a priori bounds of unknown solutions. In addition, some recent attention was on the permanence and global stability of discrete mutualism system, and many excellent results have been derived (see [6,7,8,9,10] and the references cited therein).

For the last decades, as far as the continuous and discrete multispecies Lotka-Volterra ecosystem is concerned (see [8,11,12,13,14,15,16] and the references cited therein). However, the Lotka-Volterra type models have often been severely criticized. One of the criticisms is that in such a model, the per capita rate of change of the density of each species is a linear function of densities of the interacting species. In 1973, Gilpin, Ayala et al. [17,18] claimed that more complicated competition system are needed to study qualitative properties of the systems. To this aim, they proposed several competition models. One of the models is the following competition system

\[
\dot{N}_i(t) = r_i N_i \left(1 - \frac{N_i}{K_i} - \sum_{j=1, j \neq i}^n a_{ij} \frac{N_j}{K_j}\right).
\]

Fan and Wang [19] further proposed delay Gilpin-Ayala type competition model

\[
\dot{y}_i(t) = y_i(t) \left[ r_i(t) - \sum_{j=1}^n a_{ij} y_j(t - \tau_{ij}(t))\right],
\]

\[i = 1, 2, \ldots, n.\]

By applying the coincidence degree theory, they obtained a set of easily verifiable sufficient conditions for the existence of at least one positive periodic solution of the model. Chen et al. [20] had investigated the dynamic behavior of the following discrete n-species Gilpin-Ayala competition model

\[
x_i(k + 1) = x_i(k) \exp \left( b_i(k) - \sum_{j=1}^n a_{ij}(k)(x_j(k))^\theta_{ij}\right).
\]

For general nonautonomous case, sufficient conditions which ensure the permanence and the global stability of the system are obtained; For periodic case, sufficient conditions which ensure the existence of an unique globally stable positive periodic solution of the system are obtained. Li and Chen [21] obtained that r of the species in the above system are permanent and stabilize at a unique strictly positive almost periodic solution of the corresponding subsystem, which is globally attractive, while the remaining \[n - r\] species are driven to extinction. In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (non-integral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. In fact, there have been many nice works on the positive almost periodic solutions of continuous and discrete dynamics model with almost periodic coefficients (see [5,7,8,9,10,22,23,24,25, 26,27,28] and the references cited therein).

Liao and Zhang [7] studied a discrete mutualism model with variable delays. By means of an almost periodic functional hull theory, sufficient conditions are established for the existence and uniqueness of globally attractive almost periodic solution to the system. Motivated by above, in this paper, we are concerned with the following discrete multispecies Gilpin-Ayala mutualism system.
bounded sequence 
\[ \{ \text{set of nonnegative integers} \} \text{ respectively. For any } \theta_{ij} \text{ measure the interspecific mutualism effects of the } Z_i \text{ sequences.} \]

\[ a_i(k) \text{ represent the natural growth rates of species } x_i \text{ at the } k^{\text{th}} \text{ generation, } b_i(k) \text{ are the intraspecific effects of the } k^{\text{th}} \text{ generation of species } x_i \text{ on own population, and } c_{ij}(k) \text{ measure the interspecific mutualism effects of the } k^{\text{th}} \text{ generation of species } x_j \text{ on species } x_i(i, j = 1, 2, \ldots, n, i \neq j), d_{ij}(k) \text{ are positive control sequences. } \theta_{ij} \text{ and } \theta_{ij} \text{ are positive constants.} \]

Denote as \( Z \) and \( Z^+ \) the set of integers and the set of nonnegative integers, respectively. For any bounded sequence \( \{ g(n) \} \) defined on \( Z \), define

\[ g^{u} = \sup_{n \in Z} g(n), g^{l} = \inf_{n \in Z} g(n). \]

Throughout this paper, we assume that:

\( \{ a_i(k), b_i(k), c_{ij}(k) \} \) and \( \{ d_{ij}(k) \} \) are bounded nonnegative almost periodic sequences such that

\[
0 < a^u_i \leq a_i(k) \leq a^l_i, \quad 0 < b^u_i \leq b_i(k) \leq b^l_i, \quad 0 < c^u_{ij} \leq c_{ij}(k) \leq c^l_{ij}, \quad 0 < d^u_{ij} \leq d_{ij}(k) \leq d^l_{ij}.
\]

From the point of view of biology, in the sequel, we assume that \( x(0) = (x_1(0), x_2(0), \ldots, x_n(0)) > 0 \). Then it is easy to see that, for given \( x(0) > 0 \), the system (1.1) has a positive sequence solution \( x(k) = (x_1(k), x_2(k), \ldots, x_n(k))(k \in Z^+) \) passing through \( x(0) \).

To the best of our knowledge, this is the first paper to investigate the global stability of positive almost periodic solution of discrete multispecies Gilpin-Ayala mutualism system. The aim of this paper is to obtain sufficient conditions for the existence of a unique globally attractive and uniformly asymptotically stable almost periodic solution of system (1.1) with condition (H1), by utilizing the theory of difference equation and constructing a suitable Lyapunov function and applying the analysis technique of papers [9,10,20,22,29].

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In Section 3, by applying the theory of difference inequality, we present the permanence results for system (1.1). In Section 4, we establish the sufficient conditions for the existence of a unique globally attractive and uniformly asymptotically stable almost periodic solution of system (1.1). The main results are illustrated by two examples with numerical simulation in Section 5. Finally, the conclusion ends with brief remarks in the last section.

\[ x_i(k + 1) = x_i(k) \exp \left\{ a_i(k) - b_i(k)(x_i(k))^{\theta_{ii}} + \sum_{j=1, j \neq i}^{n} c_{ij}(k) \frac{(x_j(k))^{\theta_{ij}}}{d_{ij}(k) + (x_j(k))^{\theta_{ij}}} \right\}, \quad (1.1) \]

2 Preliminaries

Firstly, we give the definitions of the terminologies involved.

Definition 2.1[(30)] A sequence \( x : Z \to R \) is called an almost periodic sequence if the \( \varepsilon \)-translation set of \( x \)

\[ E_{\varepsilon}(x, \tau) = \{ \tau \in Z : x(n+\tau) = x(n) - \varepsilon, \forall n \in Z \} \]

is relatively dense in \( Z \) for all \( \varepsilon > 0 \); that is, for any given \( \varepsilon > 0 \), there exists an integer \( l(\varepsilon) > 0 \) such that each interval of length \( l(\varepsilon) \) contains an integer \( \tau \in E_{\varepsilon}(x, \tau) \) with

\[ |x(n + \tau) - x(n)| < \varepsilon, \quad \forall n \in Z. \]

\( \tau \) is called an \( \varepsilon \)-translation number of \( x(n) \).

Definition 2.2[(31)] A sequence \( x : Z^+ \to R \) is called an asymptotically almost periodic sequence if

\[ x(n) = p(n) + q(n), \quad \forall n \in Z^+, \]

where \( p(n) \) is an almost periodic sequence and \( \lim_{n \to \infty} q(n) = 0 \).

Definition 2.3[(32)] A solution \( (x_1(k), x_2(k), \ldots, x_n(k)) \) of system (1.1) is said to be globally attractive if for any other solution \( (x_1^*(k), x_2^*(k), \ldots, x_n^*(k)) \) of system (1.1), we have

\[ \lim_{k \to \infty} (x_i^*(k) - x_i(k)) = 0, \quad i = 1, 2, \ldots, n. \]

Now, we present some results which will play an important role in the proof of the main results.
Lemma 2.1([33]). If \( \{x(n)\} \) is an almost periodic sequence, then \( \{x(n)\} \) is bounded.

Lemma 2.2([34]). \( \{x(n)\} \) is an almost periodic sequence if and only if, for any sequence \( m_i \subset Z \), there exists a subsequence \( \{m_{i_k}\} \subset \{m_i\} \) such that the sequence \( \{x(n + m_{i_k})\} \) converges uniformly for all \( n \in Z \) as \( k \to \infty \). Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.3([31]). \( \{x(n)\} \) is an asymptotically almost periodic sequence if and only if, for any sequence \( m_i \subset Z \) satisfying \( m_i > 0 \) and \( m_i \to \infty \) as \( i \to \infty \) there exists a subsequence \( \{m_{i_k}\} \subset \{m_i\} \) such that the sequence \( \{x(n + m_{i_k})\} \) converges uniformly for all \( n \in Z^+ \) as \( k \to \infty \).

Lemma 2.4([33]). Suppose that \( \{p_1(n)\} \) and \( \{p_2(n)\} \) are almost periodic real sequences. Then \( \{p_1(n) + p_2(n)\} \) and \( \{p_1(n)p_2(n)\} \) are almost periodic; \( \frac{1}{p_1(n)} \) is also almost periodic provided that \( p_1(n) \neq 0 \) for all \( n \in Z \). Moreover, if \( \varepsilon > 0 \) is an arbitrary real number, then there exists a relatively dense set that is \( \varepsilon \)-almost periodic common to \( \{p_1(n)\} \) and \( \{p_2(n)\} \).

Lemma 2.5([20]). Assume that sequence \( \{x(n)\} \) satisfies
\[
x(n + 1) \leq x(n) \exp\{a(n) - b(n)x^n(n)\} \tag{2.1}
\]
for \( n \in N \), where \( a(n) \) and \( b(n) \) are non-negative sequences bounded above and below by positive constants, \( \alpha \) is a positive constant. Then
\[
\limsup_{n \to +\infty} x(n) \leq \left( \frac{1}{\alpha b!} \right)^{\frac{1}{\alpha}} \exp\{a^n - \frac{1}{\alpha}\}. \tag{2.2}
\]

Lemma 2.6([20]). Assume that sequence \( \{x(n)\} \) satisfies
\[
x(n + 1) \geq x(n) \exp\{a(n) - b(n)x^n(n)\}, \quad n \geq N_0, \tag{2.3}
\]
\[
\limsup_{k \to +\infty} x(k) \leq x^* \quad \text{and} \quad x(N_0) > 0, \quad \text{where} \quad a(n) \quad \text{and} \quad b(n) \quad \text{are non-negative sequences bounded above and below by positive constants,} \quad \alpha \quad \text{is a positive constant and} \quad N_0 \in N. \quad \text{Then}
\]
\[
\liminf_{n \to +\infty} x(n) \geq \left( \frac{a!}{b^k} \right)^{\frac{1}{\alpha}} \exp\{a^k - b^*x^n\}. \tag{2.3}
\]

Consider the following almost periodic difference system:
\[
x(n + 1) = f(n, x(n)), \quad n \in Z^+, \tag{2.4}
\]
where \( f : Z^+ \times S_B \to R^k, S_B = \{x \in R^k : \|x\| < B\} \), and \( f(n, x) \) is almost periodic in \( n \) uniformly for \( x \in S_B \) and is continuous in \( x \). The product system of (2.1) is the following system:
\[
x(n + 1) = f(n, x(n)), \quad y(n + 1) = f(n, y(n)). \tag{2.5}
\]

and Zhang([34,35]) obtained the following Theorem.

Theorem 2.7([34,35]). Suppose that there exists a Lyapunov function \( V(n, x, y) \) defined for \( n \in Z^+ \), \( \|x\| < B, \|y\| < B \) satisfying the following conditions:

(i) \( a\|x - y\| \leq V(n, x, y) \leq b\|x - y\| \),
where \( a, b \in K \) with \( K = \{a \in C(R^+, R^+) : a(0) = 0 \) and \( a \) is increasing\};

(ii) \( \|V(n, x_1, y_1) - V(n, x_2, y_2)\| \leq L\|x_1 - x_2\| + \|y_1 - y_2\| \), where \( L > 0 \) is a constant;

(iii) \( \Delta V(n, x, y) \leq -\alpha V(n, x, y) \), where \( 0 < \alpha < 1 \) is a constant, and
\[
\Delta V(n, x, y) \equiv V(n + 1, f(n, x), f(n, y)) - V(n, x,y). \tag{3.1}
\]

Moreover, if there exists a solution \( \varphi(n) \) of (2.4) such that \( \|\varphi(n)\| \leq B^* < B \) for \( n \in Z^+ \), then there exists a unique uniformly asymptotically stable almost periodic solution \( \rho(n) \) of system (2.4) which is bounded by \( B^* \). In particular, if \( f(n, x) \) is periodic of period \( \omega \), then there exists a unique uniformly asymptotically stable periodic solution of system (2.4) of period \( \omega \).

3 Permanence

In this section, we establish a permanence result for system (1.1), which can be found by Lemma 2.5 and 2.6.

Proposition 3.1 Assume that (H1) holds. Then any positive solution \( (x_1(k), x_2(k), \ldots, x_n(k)) \) of system (1.1) satisfies
\[
m_i \leq \liminf_{k \to +\infty} x_i(k) \leq \limsup_{k \to +\infty} x_i(k) \leq M_i, \tag{3.1}
\]
The main results of this paper concern the global stability of almost periodic solution of system (1.1) with condition (H1).

**Proposition 3.3** Assume that (H1) holds. Then Ω ≠ ∅.

**Proof.** By the almost periodicity of \{a_i(k), b_i(k), c_{ij}(k)\} and \{d_{ij}(k)\}, there exists an integer valued sequence \{δ_p\} with \(δ_p \to ∞\) as \(p \to ∞\) such that

\[a_i(k + δ_p) \to a_i(k), \quad b_i(k + δ_p) \to b_i(k), \quad c_{ij}(k + δ_p) \to c_{ij}(k),\]

\[d_{ij}(k + δ_p) \to d_{ij}(k), \quad \text{as } p \to +∞.\]

Let \(ε\) be an arbitrary small positive number. It follows from Theorem 3.1 that there exists a positive integer \(N_0\) such that

\[m_i - ε ≤ x_i(k) ≤ M_i + ε, \quad k > N_0.\]

Write \(x_{ip}(k) = x_i(k + δ_p)\) for \(k ≥ N_0 - δ_p\) and \(p = 1, 2, · · · \). For any positive integer \(q\), it is easy to see that there exists a sequence \(\{x_{iq}(k)\} : p ≥ q\) such that the sequence \(\{x_{iq}(k)\}\) has a subsequence, denoted by \(\{x_{ip}(k)\}\) again, converging on any finite interval of \(Z\) as \(p \to ∞\). Thus we have a sequence \(\{y_i(k)\}\) such that

\[x_{ip}(k) → y_i(k) \text{ for } k ∈ Z \text{ as } p → +∞.\]

This, combining with gives us

\[x_i(k + 1 + δ_p) = x_i(k + δ_p) \exp \left\{a_i(k + δ_p) - b_i(k + δ_p)(x_i(k + δ_p))^{θ_i} + \sum_{j=1,j≠i}^{n} c_{ij}(k + δ_p) \frac{y_j(k)(x_i(k + δ_p))^{θ_j}}{d_{ij}(k + δ_p) + (y_j(k))^{θ_j}}\right\}, i = 1, 2, · · · , n\]

\[y_i(k + 1) = y_i(k) \exp \left\{a_i(k) - b_i(k)(y_i(k))^{θ_i} + \sum_{j=1,j≠i}^{n} c_{ij}(k) \frac{y_j(k)(y_i(k))^{θ_j}}{d_{ij}(k) + (y_j(k))^{θ_j}}\right\}, i = 1, 2, · · · , n\]

We can easily see that \(\{y_i(k)\}\) is a solution of system (1.1) and \(m_i - ε ≤ y_i(k) ≤ M_i + ε\) for \(k ∈ Z\). Since \(ε\) is an arbitrary small positive number, it follows that \(m_i ≤ y_i(k) ≤ M_i\) and hence we complete the proof.

### 4 Stability of almost periodic solution

The main results of this paper concern the global stability of almost periodic solution of system (1.1) with condition (H1).
Theorem 4.1 Assume that (H1) and

\[(H2) \quad \rho_i = \max\{|1 - \theta_i b_i^k m_i^{\theta_i^k}|, |1 - \theta_i b_i^k M_i^{\theta_i^k}|\} + \sum_{j=1, j \neq i}^n \frac{\theta_{ij} c_{ij}^k M_j^{\theta_j^k}}{d_{ij}^k} < 1, \quad i = 1, 2, \ldots, n,\]

hold. Then any positive solution \((x_1(k), x_2(k), \ldots, x_n(k))\) of system (1.1) is globally attractive.

**Proof.** Assume that \((p_1(k), p_2(k), \ldots, p_n(k))\) is a solution of system (1.1) satisfying (H1). Let

\[x_i(k) = p_i(k) \exp\{u_i(k)\}, \quad i = 1, 2, \ldots, n.\]

Then system (1.1) is equivalent to

\[
u_i(k + 1) = \ln x_i(k + 1) - \ln p_i(k + 1) = \ln x_i(k) + a_i(k) - b_i(k)(x_i(k))^{\theta_i^k} + \sum_{j=1, j \neq i}^n c_{ij}(k) \frac{(x_j(k))^{\theta_j^k}}{d_{ij}(k) + (x_j(k))^{\theta_j^k}} - \ln p_i(k) - a_i(k) + b_i(k)(p_i(k))^{\theta_i^k} - \sum_{j=1, j \neq i}^n c_{ij}(k) \frac{(p_j(k))^{\theta_j^k}}{d_{ij}(k) + (p_j(k))^{\theta_j^k}} = u_i(k) - b_i(k)[(x_i(k))^{\theta_i^k} - (p_i(k))^{\theta_i^k}] + \sum_{j=1, j \neq i}^n d_{ij}(k)c_{ij}(k)[(x_j(k))^{\theta_j^k} - (p_j(k))^{\theta_j^k}] + \sum_{j=1, j \neq i}^n \frac{d_{ij}(k)c_{ij}(k)(p_i(k))^{\theta_i^k}[(\exp\{u_i(k)\})^{\theta_i^k} - 1]}{[d_{ij}(k) + (x_i(k))^{\theta_i^k}][d_{ij}(k) + (p_i(k))^{\theta_i^k}]} - \ln p_i(k) - a_i(k) + b_i(k)(p_i(k))^{\theta_i^k} - \sum_{j=1, j \neq i}^n c_{ij}(k) \frac{(p_j(k))^{\theta_j^k}}{d_{ij}(k) + (p_j(k))^{\theta_j^k}}, \quad i = 1, 2, \ldots, n.\]

Therefore,

\[
u_i(k + 1) = u_i(k)(1 - \theta_i b_i^k(p_i(k) \exp\{\lambda_i(u_i(k))\})^{\theta_i^k}) + \sum_{j=1, j \neq i}^n \frac{d_{ij}(k)c_{ij}(k)(p_i(k))^{\theta_i^k}[(\exp\{u_i(k)\})^{\theta_i^k} - 1]}{[d_{ij}(k) + (x_i(k))^{\theta_i^k}][d_{ij}(k) + (p_i(k))^{\theta_i^k}]} - \ln p_i(k) - a_i(k) + b_i(k)(p_i(k))^{\theta_i^k} - \sum_{j=1, j \neq i}^n c_{ij}(k) \frac{(p_j(k))^{\theta_j^k}}{d_{ij}(k) + (p_j(k))^{\theta_j^k}}, \quad i = 1, 2, \ldots, n.\]

where \(\lambda_i(k), \overline{\lambda}_i(k) \in [0, 1]\). To complete the proof, it suffices to show that

\[
\lim_{k \to +\infty} u_i(k) = 0, \quad i = 1, 2, \ldots, n. \tag{4.2}
\]

In view of (H2), we can choose \(\varepsilon > 0\) such that

\[
\rho_i = \max\{|1 - \theta_i b_i^k(M_i - \varepsilon)^{\theta_i^k}|, |1 - \theta_i b_i^k(M_i + \varepsilon)^{\theta_i^k}|\} + \sum_{j=1, j \neq i}^n \frac{\theta_{ij} c_{ij}^k(M_j + \varepsilon)^{\theta_j^k}}{d_{ij}^k} < 1, \quad i = 1, 2, \ldots, n.
\]

Let \(\rho = \max\{\rho_i\}\), then \(\rho < 1\). According to Theorem 3.2, there exists a positive integer \(k_0 \in \mathbb{Z}^+\) such that

\[m_i - \varepsilon \leq x_i(k) \leq M_i + \varepsilon, \quad m_i - \varepsilon \leq p_i(k) \leq M_i + \varepsilon, \quad i = 1, 2, \ldots, n\]

for \(k \geq k_0\).

Notice that \(\lambda_i(k) \in [0, 1]\) implies that \(p_i(k) \exp\{\lambda_i(k)u_i(k)\}\) lies between \(p_i(k)\) and \(x_i(k)\), \(\overline{\lambda}_i(k) \in [0, 1]\) implies that \(p_i(k) \exp\{\overline{\lambda}_i(k)u_i(k)\}\) lies between \(p_i(k)\) and \(x_i(k)\). From (4.1), we get

\[
|u_i(k + 1)| \leq \max\{|1 - \theta_i b_i^k(M_i - \varepsilon)^{\theta_i^k}|, |1 - \theta_i b_i^k(M_i + \varepsilon)^{\theta_i^k}|\}|u_i(k)| + \sum_{j=1, j \neq i}^n \frac{\theta_{ij} c_{ij}^k(M_j + \varepsilon)^{\theta_j^k}}{d_{ij}^k}|u_j(k)|, \quad i = 1, 2, \ldots, n. \tag{4.3}
\]
for \( k \geq k_0 \).

In view of (4.3), we get
\[
\max_{1 \leq i \leq n} |u_i(k + 1)| \leq \rho \max_{1 \leq i \leq n} |u_i(k)|, \quad k \geq k_0.
\]

This implies
\[
\max_{1 \leq i \leq n} |u_i(k)| \leq \rho^{k-k_0} \max_{1 \leq i \leq n} |u_i(k_0)|, \quad k \geq k_0.
\]

Then (4.2) holds and we can obtain
\[
\lim_{k \to +\infty} |x_i(k) - p_i(k)| = 0, \quad i = 1, 2, \ldots, n. \tag{4.4}
\]

Therefore, positive solution \((x_1(k), x_2(k), \ldots, x_n(k))\) of system (1.1) is globally attractive. \(\square\)

**Theorem 4.2** Assume that (H1)-(H2) hold. Then system (1.1) admits a unique almost periodic solution which is globally attractive.

**Proof.** It follows from Proposition 3.3 that there exists a solution \((x_1(k), x_2(k), \ldots, x_n(k))\) of system (1.1) satisfying \(m_i \leq x_i(k) \leq M_i, k \in \mathbb{Z}^+\).

Suppose that \((x_1(k), x_2(k), \ldots, x_n(k))\) is any solution of system (1.1), then there exists an integer valued sequence \(\{k_1', k_2', \ldots, k_n'\}\) such that \((x_1(k + k_1'), x_2(k + k_2'), \ldots, x_n(k + k_n'))\) is a solution of the following system

\[
x_i(k + 1) = x_i(k) \exp \left\{ a_i(k + k_1') - b_i(k + k_2') (x_i(k))^{\theta_i} + \sum_{j=1, j \neq i}^{n} c_{ij}(k + k_3') \frac{(x_j(k))^{\theta_j}}{d_{ij}(k + k_5') + (x_j(k))^{\theta_j}} \right\}, \quad i = 1, 2, \ldots, n.
\]

From above discussion and Theorem 3.2, we have that not only \((x_1(k + k_1'), x_2(k + k_2'), \ldots, x_n(k + k_n'))\) but also \((\Delta x_1(k + k_1'), \Delta x_2(k + k_2'), \ldots, \Delta x_n(k + k_n'))\) are uniformly bounded, thus \((x_1(k + k_1'), x_2(k + k_2'), \ldots, x_n(k + k_n'))\) are uniformly bounded and equi-continuous. By Ascoli’s theorem[36], there exists a uniformly convergent subsequence \((x_1(k + k_1'), x_2(k + k_2'), \ldots, x_n(k + k_n')) \subseteq (x_1(k + k_1'), x_2(k + k_2'), \ldots, x_n(k + k_n'))\) such that for any \(\varepsilon > 0\), there exists a \(k_0(\varepsilon) > 0\) with the property that if \(m, n \geq k_0(\varepsilon)\) then

\[
|x_i(k + m) - x_i(k + n)| < \varepsilon,
\]

which shows from Lemma 2.3 that \((x_1(k), x_2(k), \ldots, x_n(k))\) is asymptotically almost periodic sequence. Thus, by Definition 2.2, we can express it as

\[
x_i(k) = p_i(k) + q_i(k),
\]

\(i = 1, 2, \ldots, n\), where \(\{p_i(k)\}\) is almost periodic in \(k \in \mathbb{Z}\) and \(q_i(k) \to 0\) as \(k \to +\infty\). In the following we show that \(\{p_1(k), p_2(k), \ldots, p_n(k)\}\) is an almost periodic solution of system (1.1).

From the properties of an almost periodic sequence, there exists an integer valued sequence \(\{\delta_p\}\), \(\delta_p \to +\infty\) as \(p \to +\infty\), such that

\[
a_i(k + \delta_p) \to a_i(k), \quad b_i(k + \delta_p) \to b_i(k), \quad c_{ij}(k + \delta_p) \to c_{ij}(k), \quad d_{ij}(k + \delta_p) \to d_{ij}(k), \quad as \ p \to +\infty.
\]
It is easy to know that \( x_i(k + \delta_p) \to p_i(k), \) as \( p \to \infty, \) then we have

\[
p_i(k + 1) = \lim_{p \to \infty} x_i(k + 1 + \delta_p) = \lim_{p \to \infty} x_i(k + \delta_p) \exp \left\{ a_i(k + \delta_p) - b_i(k + \delta_p)(x_i(k + \delta_p))^{\theta_{ii}} \right\}
+ \sum_{j=1, j \neq i}^{n} c_{ij}(k + \delta_p) \frac{(x_j(k + \delta_p))^{\theta_{ij}}}{d_{ij}(k + \delta_p) + (x_j(k + \delta_p))^{\theta_{ij}}}
= \lim_{p \to \infty} p_i(k) \exp \left\{ a_i(k) - b_i(k)(p_i(k))^{\theta_{ii}} + \sum_{j=1, j \neq i}^{n} c_{ij}(k) \frac{(p_j(k))^{\theta_{ij}}}{d_{ij}(k) + (p_j(k))^{\theta_{ij}}}ight\}.
\]

This prove that \( p(k) = \{(p_1(k), p_2(k), \cdots, p_n(k))\} \) satisfied system (1.1), and \( p(k) \) is a positive almost periodic solution of system (1.1).

Now, we show that there is only one positive almost periodic solution of system (1.1). For any two positive almost periodic solutions \( (p_1(k), p_2(k), \cdots, p_n(k)) \) and \( (\tilde{z}_1(k), \tilde{z}_2(k), \cdots, \tilde{z}_n(k)) \) of system (1.1), we claim that \( p_i(k) = \tilde{z}_i(k), \) \( (i = 1, 2, \cdots, n) \) for all \( k \in \mathbb{Z}^+ \). Otherwise there must be at least one positive integer \( K^* \in \mathbb{Z}^+ \) such that \( p_i(K^*) \neq \tilde{z}_i(K^*) \) for a certain positive integer \( i, \) i.e., \( \Omega = |p_i(K^*) - \tilde{z}_i(K^*)| > 0. \) So we can easily know that

\[
\Omega = \left| \lim_{p \to +\infty} p_i(K^* + \delta_p) - \lim_{p \to +\infty} \tilde{z}_i(K^* + \delta_p) \right| = \lim_{p \to +\infty} |p_i(K^* + \delta_p) - \tilde{z}_i(K^* + \delta_p)| = \lim_{k \to +\infty} |p_i(k) - \tilde{z}_i(k)| > 0,
\]

which is a contradiction to (4.4). Thus \( p_i(k) = \tilde{z}_i(k), \) \( (i = 1, 2, \cdots, n) \) hold for \( \forall k \in \mathbb{Z}^+ \). Therefore, system (1.1) admits a unique almost periodic solution which is globally attractive. This completes the proof of Theorem 4.2.

**Remark 4.3** If \( \theta_{ii} = \theta_{ij} = 1, \) for system (1.1), the condition (H2) can be simplified. Therefore, we have the following result.

**Corollary 4.4** Let \( \theta_{ii} = \theta_{ij} = 1. \) Assume that (H1) and

\[
\rho_i = \max \{ |1 - b_i^u M_i|, |1 - b_i^u M_i| \} + \sum_{j=1, j \neq i}^{n} c_{ij}^n \frac{M_j}{d_{ij}} < 1, \quad i = 1, 2, \cdots, n,
\]

hold. Then system (1.1) admits a unique almost periodic solution which is globally attractive.

In the following, the main results concern the existence of a unique uniformly asymptotically stable almost periodic solution of system (1.1) by constructing a non-negative Lyapunov function.

**Theorem 4.5** Assume that the condition (H1) hold, moreover, \( 0 < \beta < 1, \) where

\[
\beta = \min_{1 \leq i \leq n} \{ \beta_i \},
\]

\[
\beta_i = 2 \theta_{ii} b_i^u M_i - \theta_{ii}^2 b_i^u b_i^u M_i^2 - \sum_{j=1, j \neq i}^{n} \left[ \theta_{ij}^2 c_{ij}^{u^2} + (1 + 2 \theta_{ij} b_i^u M_j) \theta_{ij} c_{ij}^{u^2} + (1 + 2 \theta_{ij} b_i^u M_j) \theta_{ij} c_{ij}^{u^2} + \sum_{l=1, l \neq i, j}^{n} \theta_{il} c_{il}^{u^2} c_{ij}^{u^2} \right],
\]

\( i = 1, 2, \cdots, n. \) Then there exists a unique uniformly asymptotically stable almost periodic solution \((x_1(k), x_2(k), \cdots, x_n(k))\) of system (1.1) which is bounded by \( \Omega \) for all \( k \in \mathbb{N}^+. \)
We assume that $Q$ is bounded and that $\|Q\| \leq B$. From Proposition 3.3, we know that system (4.5) has bounded solution $(p_1(k), p_2(k), \ldots, p_n(k))$ satisfying

$$\ln m_i \leq p_i(k) \leq \ln M_i, \quad i = 1, 2, \ldots, n, \quad k \in Z^+.$$  

Hence, $|p_i(k)| \leq A_i$, where $A_i = \max\{\ln m_i, |\ln M_i|\}, i = 1, 2, \ldots, n$.

For $X \in R^n$, we define the norm $\|X\| = \sum_{i=1}^{n} |x_i|$.

Consider the product system of system (4.5)

$$\begin{align*}
p_i(k+1) &= p_i(k) + a_i(k) - b_i(k)e^{\theta_{ij}p_j(k)} + \sum_{j=1, j \neq i}^{n} c_{ij}(k) \frac{e^{\theta_{ij}p_j(k)}}{d_{ij}(k) + e^{\theta_{ij}q_j(k)}}, \\
qu_i(k+1) &= q_i(k) + a_i(k) - b_i(k)e^{\theta_{ij}q_j(k)} + \sum_{j=1, j \neq i}^{n} c_{ij}(k) \frac{e^{\theta_{ij}q_j(k)}}{d_{ij}(k) + e^{\theta_{ij}q_j(k)}}, \quad i = 1, 2, \ldots, n.
\end{align*}$$

(4.6)

We assume that $Q = (p_1(k), p_2(k), \ldots, p_n(k)), W = (q_1(k), q_2(k), \ldots, q_n(k))$ are any two solutions of system (4.5) defined on $Z^+ \times S^*$; then, $\|Q\| \leq B, \|W\| \leq B$, where $B = \sum_{i=1}^{n} A_i, S^* = \{(p_1(k), p_2(k), \ldots, p_n(k)| \ln m_i \leq p_i(n) \leq \ln M_i, i = 1, 2, \ldots, n, k \in Z^+\}$.

Let us construct a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(k, Q, W) = \sum_{i=1}^{n} (p_i(k) - q_i(k))^2.$$  

It is obvious that the norm $\|Q - W\| = \sum_{i=1}^{n} |p_i(k) - q_i(k)|$ is equivalent to $\|Q - W\| = \left(\sum_{i=1}^{n} (p_i(k) - q_i(k))^2\right)^{1/2}$; that is, there are two constants $c_1 > 0, c_2 > 0$, such that

$$c_1 \|Q - W\| \leq \|Q - W\| \leq c_2 \|Q - W\|,$$

then,

$$(c_1 \|Q - W\|)^2 \leq V(k, Q, W) \leq (c_2 \|Q - W\|)^2.$$  

Let $\psi, \varphi \in C(R^+, R^+), \psi(x) = c_1^2 x^2, \varphi(x) = c_2^2 x^2$; then, condition (i) of Theorem 2.7 is satisfied.
Finally, calculating the condition (ii) of Theorem 2.7 is satisfied.

$$(\Delta = \Delta = \Delta = \Delta)$$

where

$$\sum_{i=1}^{n} (p_i(k) - q_i(k))^2 - (\text{pr}(k) - \text{qr}(k))^2$$

$$= \sum_{i=1}^{n} [(p_i(k) - q_i(k)) + (\text{pr}(k) - \text{qr}(k))] [(p_i(k) - q_i(k)) - (\text{pr}(k) - \text{qr}(k))]$$

$$\leq \sum_{i=1}^{n} (|p_i(k) - q_i(k)| + |\text{pr}(k) - \text{qr}(k)|) (|p_i(k) - \text{pr}(k)| + |q_i(k) - \text{qr}(k)|)$$

$$\leq L \left[ \sum_{i=1}^{n} |p_i(k) - \text{pr}(k)| + \sum_{i=1}^{n} |q_i(k) - \text{qr}(k)| \right]$$

$$= L (||Q - \overline{Q}|| + ||W - \overline{W}||).$$

where $\overline{Q} = (\text{pr}(k), \text{pr}(k), \ldots, \text{pr}(k)), \overline{W} = (\text{qr}(k), \text{qr}(k), \ldots, \text{qr}(k))$, and $L = 4 \max_{1 \leq i \leq n} \{ A_i \}$. Thus, condition (ii) of Theorem 2.7 is satisfied.

Finally, calculating the $\Delta V(k)$ of $V(k)$ along the solutions of system (4.6), we have

$$\Delta V_{(4.6)}(k) = V(k+1) - V(k)$$

$$= \sum_{i=1}^{n} (p_i(k+1) - q_i(k+1))^2 - \sum_{i=1}^{n} (p_i(k) - q_i(k))^2$$

$$= \sum_{i=1}^{n} [(p_i(k+1) - q_i(k+1))^2 - (p_i(k) - q_i(k))^2]$$

$$= \sum_{i=1}^{n} \left\{ \left( p_i(k) - q_i(k) \right) - b_i(k) (e^{\theta_i p_i(k)} - e^{\theta_i q_i(k)}) + \sum_{j=1, j \neq i}^{n} \frac{c_{ij}(k) d_{ij}(k) (e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \right\}^2$$

$$- (p_i(k) - q_i(k))^2$$

$$= \sum_{i=1}^{n} \left\{ -2b_i(k)(p_i(k) - q_i(k))(e^{\theta_i p_i(k)} - e^{\theta_i q_i(k)}) + b_i^2(k) (e^{\theta_i p_i(k)} - e^{\theta_i q_i(k)})^2 \right.\left. - 2b_i(k)(e^{\theta_i p_i(k)} - e^{\theta_i q_i(k)}) \sum_{j=1, j \neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \right\}$$

$$+ \left( \sum_{j=1, j \neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \right)^2$$

$$+ 2(p_i(k) - q_i(k)) \sum_{j=1, j \neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \right\}.$$
where $\xi_{i,j}(k)$ lies between $e^{\theta_{i,j}P_i(k)}$ and $e^{\theta_{i,j}q_i(k)}$. Then, we have

$$
\Delta V_{(4.6)}(k) = \sum_{i=1}^{n} \left\{ -2\theta_{i,k}b_i(k)\xi_{i,k}(k)(p_i(k) - q_i(k))^2 + \theta^2_{i,k}b^2_i(k)\xi^2_{i,k}(k)(p_i(k) - q_i(k))^2 
- 2\theta_{i,k}b_i(k)\xi_{i,k}(k)(p_i(k) - q_i(k)) \sum_{j=1, j \neq i}^{n} \frac{\theta_{i,j}c_{i,j}(k)d_{i,j}(k)\xi_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} 
+ \left( \sum_{j=1, j \neq i}^{n} \frac{\theta_{i,j}c_{i,j}(k)d_{i,j}(k)\xi_{i,j}(k)(p_j(k) - q_j(k))}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \right)^2 
+ 2(p_i(k) - q_i(k)) \sum_{j=1, j \neq i}^{n} \frac{\theta_{i,j}c_{i,j}(k)d_{i,j}(k)\xi_{i,j}(k)(p_j(k) - q_j(k))}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \right\} 
= \sum_{i=1}^{n} \left\{ \left( -2\theta_{i,k}b_i(k)\xi_{i,k}(k) + \theta^2_{i,k}b^2_i(k)\xi^2_{i,k}(k) 
+ \sum_{j=1, j \neq i}^{n} \frac{\theta^2_{i,j}c^2_{i,j}(k)d^2_{i,j}(k)\xi^2_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \right)(p_i(k) - q_i(k))^2 
+ 2 \sum_{j=1, j \neq i}^{n} \left( \frac{1 - 2\theta_{i,j}b_j(k)\xi_{j,k}(k)}{d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \right) \frac{\theta_{i,j}c_{i,j}(k)d_{i,j}(k)\xi_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \times 
(p_i(k) - q_i(k))(p_j(k) - q_j(k)) \right\} 
\leq \sum_{i=1}^{n} \left\{ \left( -2\theta_{i,k}b_i(k)\xi_{i,k}(k) + \theta^2_{i,k}b^2_i(k)\xi^2_{i,k}(k) 
+ \sum_{j=1, j \neq i}^{n} \frac{\theta^2_{i,j}c^2_{i,j}(k)d^2_{i,j}(k)\xi^2_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \right)(p_i(k) - q_i(k))^2 
+ 2 \sum_{j=1, j \neq i}^{n} \left( \frac{1 - 2\theta_{i,j}b_j(k)\xi_{j,k}(k)}{d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \right) \frac{\theta_{i,j}c_{i,j}(k)d_{i,j}(k)\xi_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \times 
(p_i(k) - q_i(k))(p_j(k) - q_j(k)) \right\} \right\}.
$$

Then, we have

$$
\Delta V_{(4.2)}(k) \leq \sum_{i=1}^{n} [V_{i1}(k) + V_{i2}(k)],
$$

where

$$
V_{i1}(k) = -2\theta_{i,k}b_i(k)\xi_{i,k}(k) + \theta^2_{i,k}b^2_i(k)\xi^2_{i,k}(k) + \sum_{j=1, j \neq i}^{n} \frac{\theta^2_{i,j}c^2_{i,j}(k)d^2_{i,j}(k)\xi^2_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \times (p_i(k) - q_i(k))^2 
\leq \left( -2\theta_{i,k}b_i(k) + \theta^2_{i,k}b^2_i M^2 \right) + \sum_{j=1, j \neq i}^{n} \frac{\theta^2_{i,j}c^2_{i,j}(k)d^2_{i,j}(k)\xi^2_{i,j}(k)}{(d_{i,j}(k) + e^{\theta_{i,j}P_i(k)})^2(d_{i,j}(k) + e^{\theta_{i,j}q_i(k)})^2} \times (p_i(k) - q_i(k))^2.
$$
\[ V_{ij}(k) = 2 \left( \sum_{j=1,j \neq i}^{n} \frac{(1 - 2 \theta_{ij} b_{ij}(k) \xi_{ij}(k)) \theta_{ij} c_{ij}(k) d_{ij}(k) \xi_{ij}(k)}{d_{ij}(k) + e^{\theta_{ij} P_{ij}(k)}(d_{ij}(k) + e^{\theta_{ij} q_{ij}(k)})} \right) \]

\[ + \frac{1}{2} \left( \sum_{l=1}^{n} \frac{\theta_{ij} c_{ij}(k) d_{ij}(k) \xi_{ij}(k)}{d_{ij}(k) + e^{\theta_{ij} q_{ij}(k)}(d_{ij}(k) + e^{\theta_{ij} q_{ij}(k)})} \right) \times \]

\[ (p_i(k) - q_i(k))(p_j(k) - q_j(k)) \]

\[ \leq \sum_{j=1,j \neq i}^{n} \left( (1 + 2 \theta_{ij} b_{ij}^n M_{ij}) \theta_{ij} c_{ij}^n + \frac{1}{2} \sum_{l=1}^{n} \theta_{ij} \theta_{ij} c_{ij}^n \right) \left( (p_i(k) - q_i(k))^2 + (p_j(k) - q_j(k))^2 \right). \]

Hence, we have

\[ \Delta V_{(4.6)}(k) \leq \sum_{i=1}^{n} \left\{ \left( -2 \theta_{ii} b_{ii}^n + \theta_{ii}^2 b_{ii}^{n2} M_{ii}^2 \right) \right. \]

\[ + \sum_{j=1,j \neq i}^{n} \left[ \theta_{ij} c_{ij}^n + (1 + 2 \theta_{ij} b_{ij}^n M_{ij}) \theta_{ij} c_{ij}^n + \frac{1}{2} \sum_{l=1}^{n} \theta_{ij} \theta_{ij} c_{ij}^n \right] \left( (p_i(k) - q_i(k))^2 \right) \]

\[ + \sum_{j=1,j \neq i}^{n} \left( (1 + 2 \theta_{ij} b_{ij}^n M_{ij}) \theta_{ij} c_{ij}^n + \frac{1}{2} \sum_{l=1}^{n} \theta_{ij} \theta_{ij} c_{ij}^n \right) \left( (p_j(k) - q_j(k))^2 \right) \]

\[ = \sum_{i=1}^{n} \left\{ \left( -2 \theta_{ii} b_{ii}^n + \theta_{ii}^2 b_{ii}^{n2} M_{ii}^2 \right) \right. \]

\[ + \sum_{j=1,j \neq i}^{n} \left[ \theta_{ij} c_{ij}^n + (1 + 2 \theta_{ij} b_{ij}^n M_{ij}) \theta_{ij} c_{ij}^n + \frac{1}{2} \sum_{l=1}^{n} \theta_{ij} \theta_{ij} c_{ij}^n \right] \left( (p_i(k) - q_i(k))^2 \right) \]

\[ \leq -\sum_{i=1}^{n} \beta_i(p_i(k) - q_i(k))^2 \]

\[ \leq -\beta \sum_{i=1}^{n} (p_i(k) - q_i(k))^2 \]

\[ = -\beta V(k, Q, W), \]

where \( \beta = \min_{1 \leq i \leq n} \{ \beta_i \} \). That is, there exists a positive constant \( 0 < \beta < 1 \) such that

\[ \Delta V_{(4.6)}(k, Q, W) \leq -\beta V(k, Q, W). \]
From $0 < \beta < 1$, the condition (iii) of Theorem 2.7 is satisfied. So, according to Theorem 2.7, there exists a unique uniformly asymptotically stable almost periodic solution $(p_1(k), p_2(k), \ldots, p_n(k))$ of (4.5) which is bounded by $S^*$ for all $k \in \mathbb{Z}^+$. It means that there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \ldots, x_n(k))$ of (1.1) which is bounded by $\Omega$ for all $k \in \mathbb{Z}^+$. This completed the proof.

**Remark 4.6** If $n = 2$, the conditions of Theorem 4.5 can be simplified. Therefore, we have the following result.

**Corollary 4.7** Let $n = 2$, and assume further that $0 < \beta < 1$, where

$$
\beta = \min\{\beta_{12}, \beta_{21}\}, \\
\beta_{ij} = 2\theta_i b^i_j c_j \left(\frac{1}{2} M_j^i - (1 + 2\theta_i b^i_j M_i)\theta_j c^j_i - (1 + \theta_j c^j_i + 2\theta_j b^j_i M_j)\theta_i c^i_j, \right)
$$

$i, j = 1, 2, j \neq i$. Then system (1.1) admits a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k))$ which is bounded by $\Omega$ for all $k \in \mathbb{Z}^+$.

## 5 Numerical Simulations

In this section, we give the following examples to check the feasibility of our results.

**Example 5.1** Consider the discrete multispecies Gilpin-Ayala mutualism system:

\[
\begin{align*}
    x_1(k+1) &= x_1(k) \exp\left\{1.25 - 0.021 \sin(\sqrt{2}k) - (1.25 + 0.014 \sin(\sqrt{3}k))x_1^2(k)\right\} \\
    &\quad + (0.02 + 0.002 \cos(\sqrt{3}k))\frac{x_2^2(k)}{2 + x_2^2(k)} + (0.02 + 0.001 \cos(\sqrt{2}k))\frac{x_3^2(k)}{1 + x_3^2(k)}, \\
    x_2(k+1) &= x_2(k) \exp\left\{1.17 - 0.025 \sin(\sqrt{3}k) + (0.02 + 0.003 \sin(\sqrt{3}k))\frac{x_1(k)}{1 + x_1(k)}\right\} \\
    &\quad - (1.11 + 0.015 \sin(\sqrt{5}k))\frac{x_2^2(k)}{2 + x_2^2(k)} + (0.025 + 0.002 \cos(\sqrt{2}k))\frac{x_3^2(k)}{2 + x_3^2(k)}, \\
    x_3(k+1) &= x_3(k) \exp\left\{1.12 - 0.03 \sin(\sqrt{3}k) + (0.03 + 0.0025 \cos(\sqrt{2}k))\frac{x_1^2(k)}{1 + x_1^2(k)}\right\} \\
    &\quad + (0.028 + 0.0015 \cos(\sqrt{5}k))\frac{x_2^2(k)}{2 + x_2^2(k)} - (1.16 + 0.02 \sin(\sqrt{3}k))\frac{x_3^2(k)}{2 + x_3^2(k)}.
\end{align*}
\] (5.1)

A computation shows that

$$
\rho_1 \approx 0.1051, \ \rho_2 \approx 0.0144, \ \rho_3 \approx 0.0613,
$$

that $\max\{\rho_1, \rho_2, \rho_3\} < 1$. Hence, there exists a unique globally attractive almost periodic solution of system (5.1). Our numerical simulations support our results(see Figs.1,2 and 3).
Figure 1: Dynamic behavior of the first component $x_1(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (5.1) with the initial conditions $(0.98, 1.27, 0.99)$, $(1.02, 1.2, 1.05)$ and $(1.03, 1.17, 0.93)$ for $k \in [1, 100]$, respectively.

Figure 2: Dynamic behavior of the second component $x_2(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (5.1) with the initial conditions $(0.98, 1.27, 0.99)$, $(1.02, 1.2, 1.05)$ and $(1.03, 1.17, 0.93)$ for $k \in [1, 100]$, respectively.
Example 5.2 Consider the discrete Gilpin-Ayala mutualism system:

\[
\begin{aligned}
    x_1(k+1) &= x_1(k) \exp \left\{ 1.15 - 0.01 \sin(\sqrt{2}k) - (1.16 - 0.02 \cos(\sqrt{3}k))x_1^2(k) \\
    &\quad + (0.05 - 0.002 \cos(\sqrt{3}k)) \frac{x_2^\frac{1}{3}(k)}{1 + x_2^\frac{1}{3}(k)} \right\}, \\
    x_2(k+1) &= x_2(k) \exp \left\{ 1.25 - 0.025 \sin(\sqrt{3}k) + (0.02 - 0.0025 \cos(\sqrt{5}k)) \frac{x_1^2(k)}{2 + x_1^2(k)} \\
    &\quad - (1.1 - 0.02 \sin(\sqrt{7}k))x_2^\frac{1}{3}(k) \right\}.
\end{aligned}
\]  

(5.2)

A computation shows that

\[ \beta_{12} \approx 0.0143, \quad \beta_{21} \approx 0.0632, \]

that \( \min(\beta_{12}, \beta_{21}) < 1 \). Hence, there exists a unique uniformly asymptotically stable almost periodic solution of system (5.2). Our numerical simulations support our results (see Figs. 4 and 5).
Figure 4: Dynamic behavior of the first component $x_1(k)$ of the solution $(x_1(k), x_2(k))$ to system (5.2) with the initial conditions (1.03, 1.42), (1.06, 1.51) and (0.99, 1.58) for $k \in [1, 80]$, respectively.

Figure 5: Dynamic behavior of the second component $x_2(k)$ of the solution $(x_1(k), x_2(k))$ to system (5.2) with the initial conditions (1.03, 1.42), (1.06, 1.51) and (0.99, 1.58) for $k \in [1, 80]$, respectively.
6 Conclusion

In this paper, assuming that the coefficients in system (1.1) are bounded non-negative almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive. By comparative analysis, we find that when the coefficients in system (1.1) are almost periodic, the existence of a unique almost periodic solution of system (1.1) is determined by the global attractivity of system (1.1), which implies that there is no additional condition to add.

Furthermore, for the almost periodic discrete multispecies Gilpin-Ayala mutualism system (1.1) with time delays or feedback controls, we would like to mention here the question of how to study the almost periodicity of the system and whether the existence of a unique almost periodic solution is determined by the global attractivity of the system or not. It is, in fact, a very challenging problem, and we leave it for our future work.

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COMPETING INTERESTS

The author declares that no competing interests exist.

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