A Note on Guaranteed Stable Recovery of Sparse Signal in Compressed Sensing via the RIP of Orders

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Abstract

In this paper, we shall continue a study of the CS-recovery of signals studied in [1]. Under the assumption that a $m \times n$ matrix $A$ obeys the RIP of order $s$ we decompose the space of unknown vectors into sets $M_0$, $M_1$, $\ldots$, $M_7$ defined by a bias function $p_x$ on a good location $T_0 = \{1, 2, \ldots, s\}$ and research a good condition of CS-recovery.

Keywords: Compressed sensing; restricted isometry property; sparse signal recovery.

1 Introduction

This paper introduces the theory of compressed sensing (CS). For a signal $x \in \mathbb{R}^n$, let $\|x\|_0$ be the $l_0$-norm of $x$, which is defined to be the number of nonzero coordinates, $\|x\|_1$ be the $l_1$-norm of $x$ and $\|x\|_2$ be the $l_2$-norm of $x$. Let $x$ be a sparse or nearly sparse vector. Compressed sensing aims to recover a high-dimensional signal (for example: images signal, voice signal, code signal...etc.) from only a few samples or linear measurements. The efficient recovery of sparse signals has been a very active field in applied mathematics, statistics, machine learning and signal processing. Formally, one considers the following model:

$$y = Ax + z,$$

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where $A$ is a $m \times n$ matrix ($m < n$) and $z$ is an unknown noise term.

Our goal is to reconstruct an unknown signal $x$ based on $A$ and $y$ given. Then we consider reconstructing $x$ as the solution $x^*$ to the optimization problem

$$
\min_{x} \|x\|_1, \quad \text{subject to} \quad \|y - Ax\|_2 \leq \varepsilon,
$$

where $\varepsilon$ is an upper bound on the the size of the noisy contribution.

In fact, a crucial issue is to research good conditions under which the inequality

$$
\|x - x^*\|_2 \leq C_0 \|x - x_T\|_1 + C_1 \varepsilon,
$$

for suitable constants $C_0$ and $C_1$, where $T$ is any location of $\{1, 2, \ldots, n\}$ with number $|T| = s$ of elements of $T$ and $x_T$ is the restriction of $x$ to indices in $T$. One of the most generally known condition for CS theory is the restricted isometry property (RIP) introduced by [2]. When we discuss our proposed results, it is an important notion. The RIP needs that subsets of columns of $A$ for all locations in $\{1, 2, \ldots, n\}$ behave nearly orthonormal system. In detail, a matrix $A$ satisfies the RIP of order $s$ if there exists a constant $\delta$ with $0 < \delta < 1$ such that

$$
(1 - \delta)\|a\|_2^2 \leq \|Aa\|_2^2 \leq (1 + \delta)\|a\|_2^2
$$

for all $s$-sparse vectors $a$. A vector is said to be an $s$-sparse vector if it has at most $s$ nonzero entries. The minimum $\delta$ satisfying the above restrictions is said to be the restricted isometry constant and is denoted by $\delta_s$.

Many researchers has been shown that the $l_1$ optimization can recover an unknown signal in noiseless cases and in noisy cases under various sufficient conditions on $\delta$, or $\delta_s$ when $A$ obeys the RIP. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered [3]. Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ [4]. Others, $\delta_{2s} < 0.4652$ is used in [5], $\delta_{2s} < 0.4721$ for cases such that $s$ is a multiple of 4 or $s$ is very large in [6], $\delta_{2s} < 0.4734$ for the case such that $s$ is very large in [5] and $\delta_s < 0.307$ in [7]. In a recent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.5606$ for the special case such that $n \leq 4s$ [8]. J. Ji and J. Peng have improved the sufficient condition to $\delta_s < 0.308$ [9]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_s < 0.333$ for general case [10]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_s$ in case of $k \geq \frac{4}{7} s$, in particular, $\delta_{2s} < 0.707$ [11]. By using a rescaling method, H. Inoue has obtained the sufficient conditions of $\delta_s < 0.5$ and $\delta_{2s} < 0.828$ in [12].

Recently, In [1] we have researched good conditions for the recovery of sparse signals by investigating the difference between the $l_{\infty}$-norm of $h \equiv x^* - x$ and the mean $\frac{|h_1| + |h_2| + \cdots + |h_s|}{|h_1| + |h_2| + \cdots + |h_s|}$ of $\{h_1, \ldots, h_s\}$. In more details, we considered a function $p$ on $T_0 \equiv \{1, 2, \ldots, s\}$ defined by

$$
p(r) = \frac{|h_1| + |h_2| + \cdots + |h_r|}{|h_1| + |h_2| + \cdots + |h_s|}, \quad r = 1, 2, \ldots, s,
$$

where the index of $h$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_s|$ and have shown that for $c > 1$ and $\frac{c}{p(1)}$ if $A$ obeys the RIP of order $\frac{4c}{p}$ and $\delta_{2s} < \frac{1}{\sqrt{1 + \frac{4c}{p(1)}}}$, then we have stable recovery of approximately sparse signals, where $r_c$ is a natural number such that $\frac{c}{p(r_c - 1)} < p(r_c) < \frac{c}{p(r_c)}$. If $2 \leq r_c < \frac{c}{p}$. But, the function $p$ on $T_0$ and $r_c$ depend on $x$. Furthermore $r_c$ is not easily searched. In this paper, in order to compensate for these defects, we decompose $K_s(y, A) \equiv \{x \in R^n; \|y - Ax\|_2 \leq \varepsilon\}$ into
the following subsets \{M_0, M_1, \cdots, M_7\}:

\[
M_0 = \{ x \in K_s(y, A); \quad px \left( \frac{1}{5}s \right) \leq \frac{2}{5} \}, \\
M_1 = \{ x \in K_s(y, A); \quad px \left( \frac{1}{5}s \right) > \frac{2}{5} \quad \text{and} \quad px \left( \frac{1}{4}s \right) \leq \frac{1}{2} \}, \\
\vdots \\
M_6 = \{ x \in K_s(y, A); \quad px \left( \frac{k+3}{20}s \right) > \frac{k+3}{10} \quad \text{and} \quad px \left( \frac{k+4}{20}s \right) \leq \frac{k+4}{10} \}, \quad 2 \leq k \leq 6, \\
M_7 = \{ x \in K_s(y, A); \quad px \left( \frac{1}{2}s \right) = 1 \}
\]

by dividing \( T_0 = \{1, 2, \cdots, s\} \) into \( T_0 \cap [1, \frac{s}{2}) \), \( T_0 \cap (\frac{k+3}{20} \leq \frac{k+4}{20}) \), \( k = 1, \cdots, 6 \), and \( T_0 \cap (\frac{s}{2}, s] \), and we show for any \( x \in M_k(k = 1, 2, \cdots, 7) \) that if \( A \) obeys the RIP of order \( s \) and \( \delta_s < \frac{1}{1+\sqrt(\frac{2}{s})-1} \), then the inequality (1.3) holds. We also state in Section 2 the existence of CS-solution.

## 2 CS-Solution

In this section, we discuss the existence of CS-solutions mathematically.

Let a \( m \times n \) matrix \( A (m < n) \) and a data \( y \in \mathbb{R}^n \) be given. We define closed convex subsets of \( \mathbb{R}^n \) by

\[
K_0(y, A) = \{ x \in \mathbb{R}^n; \quad y = Ax \}, \\
K_s(y, A) = \{ x \in \mathbb{R}^n; \quad \|y - Ax\|_2 \leq \varepsilon \}, \quad \varepsilon > 0.
\]

When \( K_0(y, A) \neq 0 \), that is, \( y \in \mathbb{R}^n \), then \( K_0(y, A) \) and \( K_s(y, A) \) are

\[
K_0(y, A) = x_0 + \ker A
\]

for some vector \( x_0 \in K_0(y, A) \), where \( \ker A = \{ x \in \mathbb{R}^n; \quad Ax = 0 \} \). For example, if the rank \( r(A) \) of \( A \) equals \( m \), then \( AA^* \) is invertible and \( A(A^*AA^*)^{-1}y = y \). Hence, \( A^*(AA^*)^{-1}y \in K_0(y, A) \).

Let \( y \notin \mathbb{R}^n \). Since \( AR^n \) is a closed subspace of \( \mathbb{R}^n \), there exists a unique vector \( y_0 \in \mathbb{R}^n \) such that \( \|y - y_0\|_2 = \min \{ \|y - Ax\|_2; \quad x \in \mathbb{R}^n \} \). Then \( y_0 \) is a vector in \( \mathbb{R}^n \) such that \( y - y_0 \) is a vector in the orthogonal complement \((A^*AR^n)^{-1}\) of \( AR^n \). It is clear that \( K_s(y, A) \neq 0 \) if and only if \( \|y - y_0\|_2 \leq \varepsilon \). In this paper, we assume that \( K_0(y, A) \neq 0 \) in noiseless cases and \( K_s(y, A) \neq 0 \) in noise cases. We show the existence of CS-solutions.

For any \( t > 0 \) we put

\[
D_t = \{ x \in \mathbb{R}^n; \quad \|x\|_1 \leq t \}.
\]

Then \( AD_t \) is a closed convex subset of \( AR^n \) such that \( A(\partial D_t) = \partial AD_t \), where \( \partial K \) is a boundary of a set \( K \). Assume that \( y_0 \notin AD_t \). Then there exists a vector \( x_t \) in \( \partial D_t \) such that \( \|y - Ax_t\|_2 = \min \{ \|y_0 - Ax\|_2; \quad x \in D_t \} \). Since

\[
\|y - Ax_t\|_2^2 = \|y - y_0\|_2^2 + \|y_0 - Ax_t\|_2^2,
\]

we have

\[
\|y - Ax_t\|_2 = \min \{ \|y - Ax\|_2; \quad x \in D_t \}.
\]

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which implies that there exists a vector $x^*_i$ in $(x_i + \ker A) \cap D_i$ such that
\[
\|x^*_i\|_1 \leq \|x_i + x\|_1, \quad \forall x \in \ker A.
\]
Thus we have the following:

**Proposition 2.1.** Suppose that $K_i(y, A) \neq \emptyset$. Then there exists a positive number $t_0$ such that
\[
\|y_0 - Ax_{t_0}\|_2^2 = \varepsilon^2 - \|y - y_0\|_2^2
\]
and the vector $x_{t_0}$ determined by $x_{t_0}$ equals the CS-solution $x^*$. In particular, in noiseless cases, $x^* = x_{t_0}$, where $t_0$ is a positive number satisfying $y_0 = Ax_{t_0}$.

### 3 Recovery of CS

Take an arbitrary $x \in K_i(y, A)$. We denote by $x^T$ a vector obtained by changing coefficients of $x$ as follows;
\[
|h_1| \geq |h_2| \geq \cdots \geq |h_n|,
\]
where $h = (h_1, h_2, \cdots h_n) \equiv x^* - x^T$. Let $T_0 = \{1, 2, \cdots, s\}$ and we define a function $p_x(r)$ on $T_0$ depending on $x$ by
\[
p_x(r) = \frac{|h_1| + |h_2| + \cdots + |h_r|}{\|h_{T_0}\|_1}, \quad r \in T_0.
\]
By dividing $T_0 = \{1, 2, \cdots, s\}$ into $T_0 \cap [1, \frac{2}{5}], T_0 \cap (\frac{k+3}{20}, \frac{k+4}{20}] (k = 1, \cdots, 6)$ and $T_0 \cap (\frac{1}{5}, s]$, we decomposed $K_i(y, A)$ into the following subsets $\{M_0, M_1, \cdots, M_7\};$
\[
\begin{align*}
M_0 &= \{ x \in K_i(y, A); \quad p_x\left(\frac{1}{5}\right) \leq \frac{2}{5} \}, \\
M_1 &= \{ x \in K_i(y, A); \quad p_x\left(\frac{1}{5}\right) > \frac{2}{5} \text{ and } p_x\left(\frac{1}{3}\right) \leq \frac{1}{2} \}, \\
M_2 &= \{ x \in K_i(y, A); \quad p_x\left(\frac{k+3}{20}\right) > \frac{k+3}{10} \text{ and } p_x\left(\frac{k+4}{20}\right) \leq \frac{k+4}{10} \}, \quad 2 \leq k \leq 6, \\
M_7 &= \{ x \in K_i(y, A); \quad p_x\left(\frac{1}{2}\right) = 1 \}.
\end{align*}
\]
Then, $K_i(y, A) = \bigcup_{k=0}^{7} M_k$ and $M_i \cap M_j = \emptyset (i \neq j)$. (Figure 1)

Using the function $p_x(r)$ on $T_0$, we obtain a similar result to that of ([1] Theorem 2.1):

**Theorem 3.1.** Take an arbitrary $x \in M_k (k = 1, 2, \cdots 7)$. Assume that $A$ obeys the RIP of order $s$ and $\delta_s < \frac{1}{1 + \sqrt{s/n_1}}$. Then,
\[
\|x^* - x\|_2 \leq C_0^{(k)} \|x - x_s\|_1 + C_1^{(k)} \varepsilon,
\]
where $\mathbf{x}_s$ is a vector consisting of the $s$-large entries of $\mathbf{x}$ in magnitude and

$$C^{(k)}_0 = \frac{4\sqrt{\frac{20}{k+3}} - 1 \cdot \delta_s}{1 - \left(1 + \sqrt{\frac{20}{k+3}} - 1\right) \delta_s},$$

$$C^{(k)}_1 = \frac{2\sqrt{1 + \delta_s \sqrt{s}}}{\sqrt{\frac{k+3}{20}} \left(1 - \left(1 + \sqrt{\frac{20}{k+3}} - 1\right) \delta_s\right)}.$$

**Proof.** Take an arbitrary $\mathbf{x} \in M_k$. Let $r_k$ be a natural number such that

$$\frac{k+3}{20} s < r_k \leq \frac{k+4}{20} s$$

and

$$\frac{2}{s} (r_k - 1) < p_\mathbf{x}(r_k) \leq \frac{2}{s} r_k. \quad (3.1)$$

Then,

$$\frac{k+3}{10} < p_\mathbf{x}(r_k) \leq \frac{k+4}{10}. \quad (3.2)$$

We put

$$\alpha = \frac{\|h_{r_k}\|_1 + 2\|\mathbf{x} - \mathbf{x}_s\|_1}{s}.$$
Let $T_1 = \{1, 2, \ldots, r_2\}$ and $T_2 = \{r_2 + 1, \ldots, n\}$ be a decomposition of $\{1, 2, \ldots, n\}$. By (3.1) and (3.2) we have
\[
\|h_{T_2}\|_1 \leq \frac{p_x(r_2)}{r_2} \|h_{T_0}\|_1 \leq 2\alpha. \tag{3.3}
\]
By the definition of CS optimization (1.2), we have
\[
\|h_{T_2}\|_1 \leq \|h_{T_0}\|_1 + 2\|x - x_s\|_1. \tag{3.4}
\]
Hence it follows from (3.3) and (3.4) that
\[
\|h_{T_2}\|_1 = \|h_{T_0}\|_1 + \|h_{T_0} \cap T_2\|_1 \\
\leq \alpha s + (1 - p_x(r_2)) \|h_{T_0}\|_1 \\
\leq (2 - p_x(r_2)) \alpha s \\
\leq 2\alpha \left(1 - \frac{k + 3}{20}\right) s,
\]
which implies by [1] Lemma 1.1 and the Cai idea [4] that there exist $\{\lambda_i\}_{1 \leq i \leq N}$ and $\{u_i\}_{1 \leq i \leq N}$ such that
\[
h_{T_2} = \sum_{i=1}^{N} \lambda_i u_i.
\]
where

\[ 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{N} \lambda_i = 1, \]

\[ \text{supp } u_i \subseteq T_2, \quad |\text{supp } u_i| \leq \left( 1 - \frac{k + 3}{20} \right) s \]

\[ \|u_i\|_{\infty} \leq 2\alpha. \tag{3.5} \]

Hence we have

\[ \|u_i\|_2 \leq \|u_i\|_{\infty} \sqrt{|\text{supp } u_i|} \]

\[ \leq 2\alpha \sqrt{1 - \frac{k + 3}{20}}, \]

\[ |T_1| + |\text{supp } u_i| \leq r_k + \left( 1 - \frac{k + 3}{20} \right) s \leq s \]

and

\[ \alpha s = \|h_{T_1}\|_1 + 2\|x - x_s\|_1 \]

\[ = \frac{1}{p_s(r_k)} \|h_{T_1}\|_1 + 2\|x - x_s\|_1 \]

\[ \leq \frac{\sqrt{s}}{p_s(r_k)} \|h_{T_1}\|_2 + 2\|x - x_s\|_1 \]

\[ \leq \frac{\sqrt{s}}{2\sqrt{\frac{k + 3}{20}}} \|h_{T_1}\|_2 + 2\|x - x_s\|_1, \]

which implies since \( A \) obeys the RIP of order \( s \) that

\[ (1 - \delta_s)\|h_{T_1}\|_2^2 \leq \|Ah_{T_1}\|_2^2 \]

\[ \leq |\langle Ah_{T_1}, Ah \rangle| + |\langle Ah_{T_1}, Ah_{T_2} \rangle| \]

\[ \leq \sqrt{1 + \delta_s} \|h_{T_1}\|_2 \cdot 2\varepsilon + \sum_{i=1}^{N} \lambda_i |\langle Ah_{T_1}, Au_i \rangle| \]

\[ \leq 2\sqrt{1 + \delta_s} \|h_{T_1}\|_2 + \sum_{i=1}^{N} \lambda_i \delta_s \|h_{T_1}\|_2 \|u_i\|_2 \]

\[ \leq 2\sqrt{1 + \delta_s} \|h_{T_1}\|_2 \]

\[ + \delta_s \|h_{T_1}\|_2 \left( \frac{1}{2\sqrt{\frac{k + 3}{20}}} \|h_{T_1}\|_2 + \frac{2\sqrt{s}}{\sqrt{1 - \frac{k + 3}{20}}} \|x - x_s\|_1 \right) 2\sqrt{1 - \frac{k + 3}{20}} \]

\[ = 2\sqrt{1 + \delta_s} \|h_{T_1}\|_2 + \delta_s \frac{2\sqrt{\frac{k + 3}{20}}}{\sqrt{1 - \frac{k + 3}{20}}} \|h_{T_1}\|_2 \]

\[ + 4\delta_s \sqrt{\frac{20}{k + 3} - 1} \|x - x_s\|_1 \|h_{T_1}\|_2. \]

Since

\[ \left( 1 + \frac{20}{k + 3} - 1 \right) \delta_s < 1, \]

\[ Since \]

\[ 100x513.5]
we have
\[ \|h_{T_1}\|_2 \leq \frac{2\sqrt{1+\delta_s}s + \frac{4s}{\sqrt{s}}\sqrt{1 - \frac{k+3}{20} s\|x-x_s\|_1}}{1 - \left(1 + \frac{20}{k+s+1}\right)\delta_s}, \]
which implies that
\[ \|x - x^*\|_2 \leq \|x - x^*\|_1 \]
\[ = \|h_{T_0}\|_1 + \|h_{T_1}\|_1 \]
\[ \leq 2\|h_{T_0}\|_1 + 2\|x - x_s\|_1 \]
\[ \leq \frac{2\sqrt{1+\delta_s}\sqrt{s}}{p_{x}(x)} \left( \frac{2\sqrt{1+\delta_s}s + \frac{4s}{\sqrt{s}}\sqrt{1 - \frac{k+3}{20} s\|x-x_s\|_1}}{1 - \left(1 + \frac{20}{k+s+1}\right)\delta_s} \right) \]
\[ + 2\|x - x_s\|_1 \]
\[ = \frac{2\sqrt{1+\delta_s}\sqrt{s}}{p_{x}(x)} \left( \frac{2\sqrt{1+\delta_s}s + \frac{4s}{\sqrt{s}}\sqrt{1 - \frac{k+3}{20} s\|x-x_s\|_1}}{1 - \left(1 + \frac{20}{k+s+1}\right)\delta_s} \right) \]
\[ + \frac{4\sqrt{20}}{k+s+1}\delta_s\|x - x_s\|_1. \]
This completes the proof.

We state concretely the following case:
(i) Take an arbitrary \(x \in M_1\). If \(\delta_s < \frac{1}{3}\), then
\[ \|x^* - x\|_2 \leq \frac{8\delta_s}{1 - 3\delta_s}\|x - x_s\|_1 + \frac{2\sqrt{5}\sqrt{1 + \delta_s}\sqrt{s}}{1 - 3\delta_s} \]
(ii) Take an arbitrary \(x \in M_2\). If \(\delta_s < \frac{\sqrt{6} - 1}{2} \approx 0.366\), then
\[ \|x^* - x\|_2 \leq \frac{4\sqrt{3}\delta_s}{1 - (1 + \sqrt{3})\delta_s}\|x - x_s\|_1 + \frac{4\sqrt{1 + \delta_s}\sqrt{s}}{1 - (1 + \sqrt{3})\delta_s}, \]
(iii) Take an arbitrary \(x \in M_7\). If \(\delta_s < \frac{1}{2}\), then
\[ \|x^* - x\|_2 \leq \frac{4\delta_s}{1 - 2\delta_s}\|x - x_s\|_1 + \frac{2\sqrt{2}\sqrt{1 + \delta_s}\sqrt{s}}{1 - 2\delta_s}. \]
Though we have decomposed \(K_s(y, A)\) into \(M_k (k = 0, 1, \ldots, 7)\) in this paper, we may consider the other decompositions of \(K_s(y, A)\).

4 Conclusion
In a previous paper [1], we have discussed sufficient conditions of isometry constant \(\delta\) by investigating a bias function \(p_x\) defined by each unknown vector \(x\). In this paper, we decompose the space of unknown vectors into sets \(M_0, M_1, \ldots, M_7\) defined by the bias function \(p_x\). More precisely, when
$x$ is contained in $M_k$ ($1 \leq k \leq n$), the sufficient condition of $\delta_k$ is improved, and so this method is useful. When $x \in M_0$, the sufficient condition of $\delta_k$ is not improved by this method. We think that this method is more usable than a previous one in [1].

**Competing Interests**

The author declares that no competing interests exist.

**References**


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