Witt Groups of $\mathbb{P}^1$

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Abstract

In this paper we calculate the Witt groups of $\mathbb{P}^1$. It’s a known result, but we calculate it by another method: we use the localisation theorem of Balmer and the excision theorem of S. Gille.

Keywords: Witt group; toric variety; line bundle; filtration; divisor.

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1 Background

1.1 Witt Groups of a Shifted and Twisted Scheme

Let $X$ be a scheme which contains $\frac{1}{2}$ and $VB_X$ be the category of locally free coherent $\mathcal{O}_X$-modules, i.e. vector bundles. Let $\mathcal{L}$ be a line bundle over $X$. We define a duality

$$\star : VB_X \longrightarrow VB_X$$

$$(\mathcal{E}) \mapsto \mathcal{E}^* := \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}$$

which is the usual duality twisted by the line bundle $\mathcal{L}$. We identify naturally $\varpi : \mathcal{E} \rightarrow \mathcal{E}^{**}$. If $\mathcal{L} = \mathcal{O}_X$, then $\mathcal{E}^*$ is the usual dual and $\varpi$ is locally given by the application that maps an element $e$ of $\mathcal{E}$ to the evaluation at $e$. The triple $(VB_X, \star, \varpi)$ is an exact category with duality.
Definition 1.1. The Witt group of a scheme $X$ twisted by the line bundle $L$ is:

$$W(X, L) := W(VBX, *, \pi)$$  \hspace{1cm} (1.1)

For the particular case $L = O_X$, we denote $W(X, L) = W(X)$.

1.2 Derived Witt Group

Let $\mathcal{D}^b(VBX)$ be the derived category of bounded complexes of vector bundles. We provide this category by a twisted shifted duality which is composed by a duality functor $\omega : \mathcal{E} \rightarrow D_{L[n]} \mathcal{E}$ and functorial isomorphisms of biduality $D_{L[n]} : \mathcal{E} \rightarrow \mathcal{E} \vee \otimes L[n]$.

We represent the derived Witt group by:

$$W^n(X, L) := W(\mathcal{D}^b(VBX), D_{L[n]}, 1, \omega, )$$

Elements of $W^n(X, L)$ are isometric classes of such $(\mathcal{E}, \phi)$ with

$$\phi : \mathcal{E} \rightarrow D_{L[n]}(\mathcal{E})$$

is a symmetric isomorphism, with addition

$$[\mathcal{E}, \phi] + [\mathcal{F}, \psi] = [\mathcal{E} \oplus \mathcal{F}, (\begin{smallmatrix} \psi & 0 \\ 0 & \phi \end{smallmatrix})]$$

modulo metabolic classes, and the opposite is

$$- [\mathcal{E}, \phi] = [\mathcal{E}, - \phi]$$

Witt groups are functorial. To a morphism $f : Y \rightarrow X$, we have pullbacks

$$f^* : W^n(X, L) \rightarrow W^n(Y, f^* L)$$

$$[\mathcal{E}, \phi] \mapsto [f^* \mathcal{E}, f^* \phi]$$

We have also a multiplication (Gille-Nenashev)

$$W^n(X, L_1) \times W^n(X, L_2) \rightarrow W^{n+m}(X, L_1 \otimes L_2)$$

$$\left( [\mathcal{E}, \phi], [\mathcal{F}, \psi] \right) \mapsto [\mathcal{E} \otimes \mathcal{F}, \phi \otimes \psi]$$

This product is anticommutative:

$$[\mathcal{E} \otimes \mathcal{F}, \phi \otimes \psi] = (-1)^{nm} [\mathcal{F} \otimes \mathcal{E}, \psi \otimes \phi]$$

Theorem 1.1. (Homotopic Invariance [Balmer])

Let $\pi : X \times \mathbb{A}^1 \rightarrow X$ be the projection, and $i : X \rightarrow X \times \mathbb{A}^1$ the section $x \mapsto (x, 0)$. Then $\pi^*$ and $i^*$ are inverse isomorphisms:

$$W^n(X, L) \xrightarrow{\pi^*} W^n(X \times \mathbb{A}^1, \pi^* L) \hspace{1cm} (1.2)$$

Proof. See [1].

To a closed subset $Z \subset X$, there is a subcategory $\mathcal{D}^b_Z(VBX) \subset \mathcal{D}^b(VBX)$ of bounded complexes of vector bundles over $X$ which are exact over $U = X \setminus Z$. The Witt groups of this subcategory are denoted $W^n_Z(X, L)$. 


Theorem 1.2. (Localization [Balmer])

There is a long sequence

\[ \cdots \to W^n_Z(X,L) \xrightarrow{\text{Inclusion}} W^n(X,L) \xrightarrow{\text{Restriction}} W^n(U,L|_V) \xrightarrow{\partial} W^{n+1}_Z(X,L) \to \cdots \]  

(1.3)

when \( \partial \) is explicit. To a class in \( W^n(U,L|_V) \), we can write \([E_V,\phi_V]\) when \( \phi : E \to D_{L|m}(E) \) is a symmetric morphism of \( \mathcal{D}(BV_X) \) such that its restriction over \( U \) is an isomorphism. The mapping cone \( C(\phi) \) is exact over \( U \) and belongs to the subcategory \( \mathcal{D}_Z(VBX) \). Balmer provides \( C(\phi) \) with a symmetric isomorphism \( \psi : C(\phi) \to D_{L|m+1}(C(\phi)) \) which is unique up to an isometry, and we set \( \partial([E,\phi]) = [C(\phi),\psi] \).

Proof. See [2].

Theorem 1.3. (Excision [Gille])

If \( i : Z \hookrightarrow X \) is the inclusion of a closed subset \( Z \subset X \) with codimension \( d \), where \( Z \) and \( X \) are smooth, then there is a natural isomorphism

\[ i_* : W^n(Z,L|_Z \otimes \det N_{Z/X}) \xrightarrow{\sim} W^{n+d}_Z(X,L). \]  

(1.4)

Proof. See [3].

If \( i \) is the inclusion \( i : Z \hookrightarrow Z \times \mathbb{A}^d \) given by \( i(z) = (z,0) \), then it may be explicit. Suppose that \( x_1, x_2, \ldots, x_d \) are the standard coordinates in \( \mathbb{A}^d \), and \( K(x_1, \ldots, x_d) \) is the Koszul complex.

Theorem 1.4. Consider the inclusion \( i : Z \hookrightarrow Z \times \mathbb{A}^d \) and denote the projection \( \pi : Z \times \mathbb{A}^d \to Z \). The isomorphism of the excision theorem is:

\[ i_* : W^n(Z,L|_Z \otimes \det N_{Z/X}) \to W^{n+1}_Z(X,L). \]

Proof. See [3].

Theorem 1.5 (Balmer). The Witt groups of a point \( \text{Spec}(k) = \mathbb{A}^0_k = \mathbb{P}^0_k = \ast = pt \) are

\[ W^n(\mathbb{A}^0_k,\mathcal{O}) = \begin{cases} W(k) & \text{for } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases} \]  

(1.5)

where \( W(k) \) denotes the Witt group of isometry classes of anisotropic quadratic forms over \( k \).

Proof. See [4].

Remark 1.1. In this work, the value of \( W(k) \) is not important.

Theorem 1.6 (Walter). Let \( X \) be a scheme which contains \( \frac{1}{2} \). Consider the projective space \( \mathbb{P}^r_X \) over \( X \) such that \( r \geq 1 \). Let \( m \in \mathbb{Z}/2 \) and \( \mathcal{O}(m) \in \mathbf{Pic}(\mathbb{P}^r_X)/2 \).

If \( r \) is even, then \( W^r(\mathbb{P}^r_X,\mathcal{O}(m)) = \begin{cases} W^r(X) & \text{if } m \text{ is even} \\ W^{r-1}(X) & \text{if } m \text{ is odd} \end{cases} \)

If \( r \) is odd, then \( W^r(\mathbb{P}^r_X,\mathcal{O}(m)) = \begin{cases} W^r(X) \oplus W^{r-1}(X) & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases} \)

Proof. See [5].
1.3 Torus

Let $G_m = \mathbb{A}^1 \setminus 0$. This $G_m$ is an affine variety: $G_m = \text{Spec}(k[T,T^{-1}])$.

**Definition 1.2.** An algebraic torus is an algebraic group which is isomorphic to a finite product of $G_m$:

$$G_m \times G_m \ldots \times G_m = G^n_m.$$ 

**Theorem 1.7.** Let $x$ be the coordinate on $G_m$. For all variety $Y$, all line bundle $\mathcal{L}$ over $Y$ and all $n$ we have the isomorphism:

$$W^n(Y, \mathcal{L}) \oplus W^n(Y, \mathcal{L}) \xrightarrow{\cong} W^n(Y \times G_m, \pi^* \mathcal{L})$$

$$([(\mathcal{E}, \phi), ([\mathcal{F}, \psi]) \mapsto \left[ \pi^* \mathcal{E} \oplus \pi^* \mathcal{F}, \left( \begin{array}{cc} x & 0 \\ \pi^* \psi \\ \end{array} \right) \right].$$

We can denote that isomorphism by $(1, (x)) : (e, f) \mapsto e + (x)f$, when we identify every symmetric complex in $Y$ to its pullback into $W^n(Y \times G_m)$.

**Proof.** See [3].

**Remark 1.2.** We have a long localisation exact sequence:

$$\cdots \to W^n_{\mathcal{L}_0}(Y \times A^1, \pi^* \mathcal{L}) \xrightarrow{\pi^*} W^n(Y \times A^1, \pi^* \mathcal{L}) \xrightarrow{s_1} W^n(Y \times G_m, \pi^* \mathcal{L}) \xrightarrow{\partial} W^{n+1}_{\mathcal{L}_0}(Y \times A^1, \pi^* \mathcal{L}) \xrightarrow{\partial} \cdots$$

Where $s_0 : Y \rightarrow Y \times A^1$ is the null section and $s_1 : Y \rightarrow Y \times A^1$ is the constant section at 1.

**Lemma 1.8.** There is an isomorphism between the localisation exact sequence and the following one:

$$0 \longrightarrow W^n(Y, \mathcal{L}) \xrightarrow{i_*} W^n(Y \times G_m, \pi^* \mathcal{L}) \xrightarrow{\partial} W^{n+1}_{\mathcal{L}_0}(Y \times A^1, \pi^* \mathcal{L}) \xrightarrow{\partial} 0$$

$$0 \longrightarrow W^n(Y, \mathcal{L}) \xrightarrow{i_*} W^n(Y \times G_m, \pi^* \mathcal{L}) \xrightarrow{\partial} W^{n+1}_{\mathcal{L}_0}(Y \times A^1, \pi^* \mathcal{L}) \xrightarrow{\partial} 0$$

where $i_1$ and $p_2$ denote the inclusion of the first factor and the projection on the second one, $s_0$ the null section and finally $x$ is the coordinate on $A^1$ which vanishes at 0.

**Proof.** See [3].

**Remark 1.3.** The Witt groups of $G_m$ are known; if $x_1, x_2, \ldots, x_n$ are the coordinates on $G^n_m$, then

$$W^1(G^n_m) = W^2(G^n_m) = W^3(G^n_m) = 0.$$ 

Also we have:

$$W^0(G_m) = W(k)(1) \oplus W(k)(x),$$

and

$$W^0(G_m \times G_m) = W(k)(1) \oplus W(k)(x_1) \oplus W(k)(x_2) \oplus W(k)(x_1x_2).$$

etc.
2 Witt Groups of $\mathbb{P}^1$

Let $k$ an algebraically closed field and $\mathbb{P}^1 := \mathbb{P}^1_k$. Let $\mathcal{D}^b(\mathbb{P}^1) = \mathcal{D}^b(VB_{\mathbb{P}^1})$ the derived category of bounded complexes of vector bundles over $\mathbb{P}^1$ with the usual duality $\mathcal{E}^* = \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{E}^*, \mathcal{O}_{\mathbb{P}^1})$.

Let calculate the Witt groups of $\mathbb{P}^1$ using the localisation sequence with the closed subset $Z = \{0\} \cup \{\infty\}$ and its open complementary $\mathbb{G}_m$. Firstly we have $\text{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$. As Witt groups are periodic modulo 2 on $L \in \text{Pic}(X)$, so it really remains two kinds of groups to calculate: $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ and $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.

2.1 Calculation of $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$

Theorem 2.1. For all $n \in \mathbb{N}$,

$$\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \begin{cases} W(k) & \text{if } n \equiv 0 \text{ or } 1 \ [4], \\ 0 & \text{otherwise}. \end{cases}$$ (2.1)

Proof. We have the following exact sequence:

$$\cdots \longrightarrow \mathcal{W}^n(\mathbb{P}^1) \longrightarrow \mathcal{W}^n(\mathbb{G}_m) \longrightarrow \mathcal{W}^{n+1}(\mathbb{P}^1) \longrightarrow \cdots \tag{1}$$

$$\begin{array}{c}
\langle (1), (x) \rangle \\ \cong \end{array} \begin{array}{c}
\mathcal{W}^n(k) \oplus \mathcal{W}^n(k) \\ \mathcal{W}^{n+1}(\mathbb{P}^1) \end{array}$$

As $\mathcal{W}^n(k) = 0$ for $n \not\equiv 0 \pmod{4}$, we found $\mathcal{W}^2(\mathbb{P}^1) = 0$ and $\mathcal{W}^3(\mathbb{P}^1) = 0$, and it becomes the exact sequence:

$$0 \longrightarrow \mathcal{W}^0(\mathbb{P}^1) \longrightarrow \mathcal{W}^0(\mathbb{G}_m) \longrightarrow \mathcal{W}^n(\mathbb{P}^1) \oplus \mathcal{W}^n(\mathbb{P}^1) \longrightarrow \mathcal{W}^1(\mathbb{P}^1) \longrightarrow 0.$$ 

We can separate two connected components 0 and $\infty$.

Then we obtains

$$\partial_0(a(1) + b(x)) = i_0(a)$$

and

$$\partial_{\infty}(a(1) + b(x)) = \partial_{\infty}(a(1) + b(x^{-1})) = i_{\infty}(b)$$

because $(x) = (x^{-1})$.

Thus it grows

$$0 \longrightarrow \mathcal{W}^0(\mathbb{P}^1) \rightarrow \mathcal{W}(\mathbb{P}^1) \rightarrow \mathcal{W}(\mathbb{P}^1)$$

We define a filtration of $\mathcal{D}^b(\mathbb{P}^1)$ as

$$0 \subset \mathcal{D}^b_{(0, \infty)}(\mathbb{P}^1) \subset \mathcal{D}^b(\mathbb{P}^1).$$

That gives us a short exact sequence of categories:

$$0 \rightarrow \mathcal{D}^b_{(0, \infty)}(\mathbb{P}^1) \rightarrow \mathcal{D}^b(\mathbb{P}^1) \rightarrow \mathcal{D}^b(\mathbb{P}^1)/\mathcal{D}^b_{(0, \infty)}(\mathbb{P}^1) \rightarrow 0.$$
and
\[ \mathcal{D}^\alpha_0(P^1)/\mathcal{D}^\alpha_{(0,\infty)}(P^1) \cong \mathcal{D}^\alpha_0(P^1 \setminus \{0, \infty\}). \]

Now
\[ \mathcal{W}^p_{(0,\infty)}(P^1) \cong \mathcal{W}^p_0(P^1) \oplus \mathcal{W}^p_{(\infty)}(P^1). \]

Then with respect to the excision theorem of Gille, we obtain:
\[ \mathcal{W}^p_{(0)}(P^1) := \mathcal{W}^p_0(P^1) \cong \mathcal{W}^{p-1}(\{0\}), \]
and
\[ \mathcal{W}^p_{(\infty)}(P^1) := \mathcal{W}^p_{(\infty)}(P^1) \cong \mathcal{W}^{p-1}(\{\infty\}). \]

Thus, if \( p \equiv 1 \pmod{4} \), we have
\[ \mathcal{W}^p_{(0)}(P^1) \cong \mathcal{W}(k) \quad \text{et} \quad \mathcal{W}^p_{(\infty)}(P^1) \cong \mathcal{W}(k). \]

Recall that for \( x = \frac{q}{p} \) where \( X_0 = 0 \) at \( \{0\} \) and \( X_1 = 0 \) at \( \{\infty\} \), the isomorphism \( \mathcal{W}(k) \cong \mathcal{W}^p_{(0)}(P^1) \) is described by:
\[
(a_1, a_2, \ldots, a_r) \mapsto \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
-a_1x_1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & x_0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]
\[
\cong \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
-a_1x_1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
-x_0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & -x_0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\]

With respect to the localisation theorem of Balmer, the spectral sequence is reduced to:
\[ \cdots \to \mathcal{W}^p(\mathcal{D}^\alpha_{(0,\infty)}(P^1)) \xrightarrow{\alpha} \mathcal{W}^p(\mathcal{D}^\alpha(P^1)) \xrightarrow{\beta} \mathcal{W}^p(\mathcal{D}^\alpha_0(P^1 \setminus \{0, \infty\})) \xrightarrow{\gamma} \mathcal{W}^{p+1}(\mathcal{D}^\alpha_{(0,\infty)}(P^1)) \to \cdots, \]
where \( \alpha \) is the inclusion and \( \beta \) is the restriction.

Then for \( p = 0 \), we have:
\[ 0 \to \mathcal{W}^0(P^1) \to \mathcal{W}^0(G_m) \xrightarrow{\alpha} \mathcal{W}^0_{(0,\infty)}(P^1) \to \mathcal{W}^0_0(P^1) \to 0. \]

Recall that \( G_m = \text{Spec}(k[t, t^{-1}]) \) and \( \mathcal{W}^0(G_m) \cong \mathcal{W}(k)(1) \oplus \mathcal{W}(k)(x) \) which is a free \( \mathcal{W}(k) \)-module of rank 2.

Describe now \( \partial(1) \) and \( \partial(x) \):

\[
\begin{align*}
\cdot \quad (1) & \colon 0 \longrightarrow \mathcal{O}_{P^1} \longrightarrow 0 \quad \text{and} \quad \partial((1)) = 0 \longrightarrow \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_{P^1} \longrightarrow 0 \\
& 0 \longrightarrow \mathcal{O}_{P^1} \longrightarrow 0 \quad 0 \longrightarrow \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_{P^1} \longrightarrow 0
\end{align*}
\]

The two lines of \( \partial((1)) \) are acyclic complexes so \( \partial((1)) = 0 \), then
\[ \mathcal{W}(k)(1) \subset \ker(\partial) = \mathcal{W}^0(P^1) \]
\[ \langle x \rangle := 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0 \quad \text{and} \quad \partial(\langle x \rangle) = 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \]

\[ 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \quad \text{and} \quad \partial(\langle x \rangle) = 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{x_1} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \]

which prove that \( \partial(\langle x \rangle) = \langle 1 \rangle \).

Then \( \langle 1 \rangle \mapsto (0, 0) \) and \( \langle x \rangle \mapsto (\langle 1 \rangle, \langle 1 \rangle) \).

Next, \( \mathbb{P}^1 \) with trivial duality has the following Witt groups:

\[ W^0(\mathbb{P}^1) = \ker(\partial) = W(k)(1), \]

and

\[ W^1(\mathbb{P}^1) = \coker(\partial) = \frac{W(k) \oplus W(k)}{W(k)((1), (1))} \cong W(k). \]

\[ \square \]

### 2.2 Calculation of \( W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \)

**Theorem 2.2.** For all \( n \in \mathbb{N} \), \( W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0 \).

The groups \( W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \) are more complicated. We use the theory of divisors.

**Definition 2.1.** An irreducible divisor on a smooth variety \( X \) is an irreducible subvariety \( Z \subset X \) of codimension 1. A divisor on a smooth variety \( X \) is a formal sum of irreducible divisors with coefficients in \( \mathbb{Z} \)

\[ D = a_1Z_1 + a_2Z_2 + \cdots + a_rZ_r. \]

Divisors on \( X \) form an abelian group \( \text{Div}(X) \). A divisor is effective if all its coefficients \( a_i \geq 0 \). We write \( D \succ E \) if \( D - E \) is effective.

For an open \( U \subset X \), we have a restriction morphism

\[ \text{Div}(X) \longrightarrow \text{Div}(U) \quad D = \sum a_iZ_i \longrightarrow D_{|U} = \sum_{Z_i \cap U \neq \emptyset} a_i(Z_i \cap U) \]

To every irreducible divisor is a non-archimedean valuation \( v_Z : K(X)^* \rightarrow \mathbb{Z} \), which measures the order of cancellation or the pole order of \( f \in K(X)^* \) at the generic point of \( Z \). The principal divisor associated to a function \( f \in K(X)^* \) is \( \text{div}(f) = \sum_{Z \text{ irreducible}} v_Z(f) \).

For each divisor \( D \) we have a subsheaf \( \mathcal{O}_X(D) \) with sections on each open set \( U \subset X \) are

\[ \mathcal{O}_X(D) = \{ f \in K(X)^*/\text{div}(f)_{|U} \succ -D_{|U} \} \cup \{0\}. \]

The bundle \( \mathcal{O}_X(D) \) is the sheaf of sections of a line bundle is also noted that \( \mathcal{O}_X(D) \). The general theorem of this theory is:
Theorem 2.3. To each smooth variety $X$, it corresponds an exact sequence:

$$1 \rightarrow \mathcal{O}(X)^\times \rightarrow K(X)^\times \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\partial} \mathcal{O}_X(\mathcal{D}) \rightarrow \text{Pic}(X) \rightarrow 1.$$ 

Let denote $L_1 = \pi^* L \otimes \mathcal{O}_{Y \times \mathbb{A}^1}(s_0(Y))$. It’s a line bundle over $Y \times \mathbb{A}^1$ whose sections are rational sections of $L$ with at worst a simple pole along $s_0(Y)$ and which are regular everywhere else.

Lemma 2.4. There is an isomorphism between the localisation exact sequence and the following one:

$$0 \rightarrow W^n(Y \times \mathbb{A}^1, L_1) \xrightarrow{\text{pr}} W^n(Y \times \mathbb{G}_m, L_1) \xrightarrow{\partial} W^{n+1}_{\mathcal{O}_Y}(Y \times \mathbb{A}^1, L_1) \rightarrow 0$$

where $i_1$ and $p_1$ denote the inclusion of the first factor and the projection on the second one, $s_0$ the null section and finally $x$ is the coordinate on $\mathbb{A}^1$ which vanishes at 0.

Note that the isomorphism in middle of diagrams of this lemma and the lemma is the same $\pi^* L$ and $L_1$ have the same restrictions to $Y \times \mathbb{G}_m$, but the role of factors of the direct sum in the bottom exact sequence is reversed.

Lemma 2.5. Let $\xi : L \xrightarrow{\cong} L_1$ be an isomorphism of line bundles over a variety $X$. Then

$$\xi_2 : W^n(X, L) \xrightarrow{[E, \phi]} W^n(X, L_1)$$

is an isomorphism between derived Witt groups which is compatible with restriction to open subsets and to localisation long exact sequences.

We identify $\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(0)$. But $\mathbb{P}^1$ is the union of two open subsets $\mathbb{A}^1_0 = \text{Spec}(K[x])$ and $\mathbb{A}^1_\infty = \text{Spec}(K[x^{-1}])$. We have $\mathcal{O}_{\mathbb{P}^1}(0)(k[x]) = x^{-1}k[x]$ and $\mathcal{O}_{\mathbb{P}^1}(0)(k[x^{-1}]) = k[x^{-1}]$.

Proof of theorem 2.2. For $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, we identify $\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, all germs of rational functions with at worst a simple pole at 0 and regular elsewhere. Then the localisation sequence becomes:

$$0 \rightarrow W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow W^n(\mathbb{G}_m) \xrightarrow{\beta_0} W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow 0.$$ 

Proof of theorem 2.2. For $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, we identify $\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, all germs of rational functions with at worst a simple pole at 0 and regular elsewhere. Then the localisation sequence becomes:

$$0 \rightarrow W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow W^n(\mathbb{G}_m) \xrightarrow{\beta_0} W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow 0.$$ 

Here we have $(\beta_0, \beta_\infty) : W^n(\mathbb{G}_m) \rightarrow W(k) \oplus W(k)$, but $W^n(\mathbb{G}_m) \cong W(k)(1) \oplus W(k)(t)$. Thus $\beta_0 : a(1) + b(t) \mapsto a \beta_\infty : a(1) + b(t) \mapsto b$. Then $(\beta_0, \beta_\infty)$ is an isomorphism and its kernel is $\ker(\beta_0, \beta_\infty) = W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$, and its cokernel is $\text{coker}(\beta_0, \beta_\infty) = W^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$.

3 Conclusion

Arason proved that: if $k$ is a field of characteristic not 2 and $n \geq 1$ then $W^n(\mathbb{P}^1_k) = W(k)$. In 90’s Balmer introduced $W^n(X)$, where $X$ is a derived and more general triangulated categories, which have a lot of applications, see for example [6]. Later, Walter proved a projective bundle theorem, which allowed the calculation of $W^n(\mathbb{P}^r_x, \mathcal{O}(m))$ where $X$ is a scheme containing $\frac{1}{2}, r \geq 1, m \in \mathbb{Z}/2,$
\( \mathbb{P}_X \) is the \( r \)-projective space over \( X \) and \( \mathcal{O}(m) \in \text{Pic}(\mathbb{P}_X)/2 \) [Picard group].

In this paper, we calculate \( W^n(\mathbb{P}^2) \) using the famous Balmer’s localization sequence, a simple method which permits us to eliminate some hardness. The mentioned method opens the road to find, with real few geometric complexities, \( W^n(\mathbb{P}^2) \) and \( W^n(\mathbb{P}^3) \). That is our actual objective.

**Competing Interests**

The authors declare that no competing interests exist.

**References**


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