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Proposition of a Recursive Formula to Calculate the Higher Order Derivative of a Composite Function without Using the Resolution of the Diophantine Equation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

The formula of Faà Di Bruno provides a powerful tool to calculate the higher order derivative of a composite function. Unfortunately it has three weaknesses: it is not a recursive formula, it totally depends on the resolution of the diophantine equation and a change in the order of the derivative requires the total change of the calculation. With these weaknesses and the absence of a formula to program, Faà Di Bruno's formula is less useful for formal computation.
Other complicated techniques based on finite difference calculation (see [1]) are recursive, however the complexity of the calculation algorithm is very high. There is as well some techniques based on graphs (see [2]) to calculate the coefficients to a certain order, but without giving the general formula.
In our work we propose a new formula to calculate the higher order derivative of a composite function gof. It is of great interest, because it is recursive and it is not based on the resolution of the diophantine equation. We complete this work by giving an expression that allows to find directly the n-th derivative of a composite function.

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1 Introduction

Based on the derivation of a composite function theorem (see [3]), the higher derivatives of a composite function $(gof)^{(n)}$ can be calculated for all $n \in \mathbb{N}$ as:

$$(gof)^{'} = (g^{'}of)f^{'}$$
$$(gof)^{''} = (g^{''}of)(f^{'})^2 + (g^{'}of)f^{''}$$

The famous Francesco Faà Di Bruno's formula Published in the W.P Johnson Article (see [4]-[5],[6]) is the first formula that generalizes the theorem of deriving a composite function.

$$(gof)^{(n)} = \sum_{\sum_{i=1}^{n} ia_i = n} g^{(p)}of \frac{n!}{\prod_{i=1}^{n} a_i!(i!)^{a_i}} \times \prod_{i=1}^{n}(f^{(i)})^{a_i},\ with\ \sum_{i=1}^{n} a_i = p$$

Where:

- $\{a_i \in \mathbb{N}/i = 1, ..., n\}$, solutions of the diophantine equation $\sum_{i=1}^{n} ia_i = n$,
- $f^{(i)}$ denotes the i-th derivative of the function f.

Example 1:(see [7]) let us look for the third derivative of gof using Faà Di Bruno's formula.

The solutions of the diophantine equation $\sum_{i=1}^{3} ia_i = 3\ /a_i \in \mathbb{N}$ are : $(0, 0, 1); (1, 1, 0); (3, 0, 0)$.

So we have:

$$(gof)^{(3)} = \frac{3!}{0!0!1!}(g^{'}of)(\frac{f^{(3)}}{3!}) + \frac{3!}{1!1!0!}(g^{(2)}of)(\frac{f^{'}}{1!})(\frac{f^{(2)}}{2!}) + \frac{3!}{3!0!0!}(g^{(3)}of)(\frac{f^{'}}{1!})^3$$
$$(gof)^{(3)} = (g^{(3)}of)(f^{'})^3 + 3(g^{(2)}of)f^{'}f^{(2)} + (g^{'}of)f^{(3)}$$

2 The Weaknesses of Faà Di Bruno's Formula

Let us take back the example (1). To find the third derivative we were forced to solve the equation $\sum_{i=1}^{n} ia_i = n$, which is not always easy for a large n. Indeed what makes the formula of Faà Di Bruno less useful for the higher order derivatives is that it totally depends on the resolution of the diophantine.

A change in the order of the derivative requires the total change of the calculation, which implies the absence of a recurrent formula. These two Weaknesses justify the need to find other formulas or methods that don't suffer from these feebleness.

3 Recurcive Formula Related to the n-th Derivative

Due to this two Weaknesses of Faà Di Bruno's formula (Diophantine equation and the non-recursion), here is a full work to find other different formulas other than Faà. We mention the method based on trees in [8] and a recursive formula based on directional derivatives in [9] and [10].

In the following work we denote:

$$\Delta_n = (gof)^{(n)}$$

$$\sigma_p = (g^{(p)} of)$$
$$x_q = f^{(q)}$$

In this section we propose a recursive formula to calculate the n-th derivative of a composite function (gof). The direct calculation from the derivation formula to a certain order gives us :

$\Delta_1 = \sigma_1 x_1$

$\Delta_2 = \sigma_2 x_1^2 + \sigma_1 x_2$

$\Delta_3 = \sigma_3 x_1^3 + \sigma_2(3x_1x_2) + \sigma_1 x_3$

$\Delta_4 = \sigma_4 x_1^4 + \sigma_3(6x_1^2x_2) + \sigma_2(3x_1^2 + 4x_1x_3) + \sigma_1 x_4$

$\Delta_5 = \sigma_5 x_1^5 + \sigma_4(10x_1^3x_2) + \sigma_3(15x_1x_2 + x_1^2x_3) + \sigma_2(10x_2x_3 + 5x_1x_4) + \sigma_1 x_5$

$\Delta_6 = \sigma_6 x_1^6 + \sigma_5(15x_1^4x_2) + \sigma_4(45x_1^2x_2^2 + 20x_1^3x_3) + \sigma_3(150x_2^3 + 60x_1x_2x_3 + 15x_1^2x_4)$

$+\sigma_2(10x_3^2 + 15x_2x_4 + 6x_1x_6) + \sigma_1 x_6$

$\Delta_7 = \sigma_7 x_1^7 + \sigma_6(21x_1^5x_2) + \sigma_5(105x_1^3x_2^2 + 35x_1^4x_3) + \sigma_4(105x_2^3 + 210x_1^2x_2x_3 + 35x_1^3x_4) + \sigma_3(105x_2^2x_3 + 70x_1x_3^2 + 105x_1x_2x_4 + 21x_1^2x_5) + \sigma_2(35x_3x_4 + 21x_2x_5 + 7x_1x_6) + \sigma_1 x_7$

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$\Delta_{10} = \sigma_{10} x_1^{10} + \sigma_9(45x_1^8x_2) + \sigma_8(630x_1^6x_2^2 + 120x_1^7x_3) + \sigma_7(3150x_1^4x_2^3 + 2520x_1^5x_2x_3 + 210x_1^6x_4) + \sigma_6(4725x_1^2x_2^4 + 12600x_1^3x_2^2x_3 + 2100x_1^4x_3^2 + 3150x_1^4x_2 + 252x_1^5x_5) + \sigma_5(945x_2^5 + 12600x_1x_2^3x_3 + 12600x_1^2x_2x_3^2 + 9450x_1^2x_2^2x_4 + 4200x_1^3x_3x_4 + 2520x_1^3x_2x_5 + 210x_1^4x_6) + \sigma_4(6300x_2^2x_3^3 + 3150x_2^3x_4 + 2800x_1x_2^3 + 12600x_1x_2x_3x_4 + 3780x_1x_2^2x_5 + 1575x_1^2x_4^2 + 2520x_1^2x_3x_5 + 1260x_1^2x_2x_6 + 120x_1^3x_7) + \sigma_3(2100x_3^2x_4 + 1575x_2x_4^2 + 2520x_2x_3x_5 + 630x_2^2x_6 + 1260x_1x_4x_5 + 840x_1x_3x_6 + 360x_1x_2x_7 + 45x_1^2x_8) + \sigma_2(126x_5^2 + 210x_4x_6 + 120x_3x_7 + 45x_2x_8 + 10x_1x_9) + \sigma_1 x_{10}$

We notice that Δ_n is composed of terms , each one of them belongs to a specific family and the index σ characterizes this family.

Remark

The term Δ_n depends on $\sigma_p(A\prod_{i=1}^{n} x_i^{a_i})$, where A,$a_i$ are natural numbers, so we have :

$$\Delta_n = \sum_{p=1}^{n} \sigma_p \sum_{i=1}^{n} A \prod_{i=1}^{n} x_i^{a_i}$$

Remark

For every $\sigma_p(A\prod_{i=1}^{n} x_i^{a_i})$ of Δ_n we have :

$$\sum_{i=1}^{n} a_i = p, \quad p \geq 1$$

Proof

For $n = 1$, $\Delta_1 = \sigma_1 x_1^1$ so $\sum_{i=1}^{n} a_i = 1$. We assume that the statement is true up to order n and let us show it for $n + 1$.

Because $\Delta_{n+1} = \Delta_n^{'}$, the expression $\sigma_p(A\prod_{i=1}^{n} x_i^{a_i})$ changes in Δ_{n+1} as follow:

$$(\sigma_p(A\prod_{i=1}^{n} x_i^{a_i}))^{'} = \sigma_p^{'}(A\prod_{i=1}^{n} x_i^{a_i}) + \sigma_p(A\prod_{i=1}^{n} x_i^{a_i})^{'}$$

$$\sigma_p^{'}(A\prod_{i=1}^{n} x_i^{a_i}) = \sigma_{p+1}(A\prod_{i=1}^{n+1} x_i^{a_i})$$

With a_1 becoming $a_1 + 1$ et $a_{n+1} = 0$

$$\sum_{i=1}^{n+1} a_i = a_1 + 1 + \sum_{i=2}^{n+1} a_i = p + 1$$

By using the Leibniz formula in [11], the second term can be written as:

$$\sigma_p(A\prod_{i=1}^{n} x_i^{a_i})^{'} = \sigma_p.A\sum_{t=1}^{n}(a_t.x_t^{a_t-1} \times x_{1+t}^{1} \prod_{j=1,j\neq t}^{n} x_j^{a_j})$$

$$(a_t - 1) + 1 + \sum_{j=1,j\neq t}^{n} a_j = -1 + 1 + \sum_{j=1}^{n} a_j = p.$$

The previous proposal shows that p is implicitly indicated by the exhibitors of $x_i(\sum_{i=1}^{n} a_i = p)$.

Therefore we put:

$$T_n = \sum_{i=1}^{n} A\prod_{i=1}^{n} x_i^{a_i}$$

instead of

$$\Delta_n = \sum_{p=1}^{n} \sigma_p \sum_{i=1}^{n} A\prod_{i=1}^{n} x_i^{a_i}$$

In the two following paragraphs we present the new main results (Theorem 1 and 2). The first is a recursive formula that is not based on solving the diophantine equation to calculate a higher derivatives of a composite function $(gof)^{(n)}$. While the second allows us to calculate a higher derivatives of a composite function directly.

4 Recursive Formula to Calculate $(gof)^{(n)}$

The first term of T_n has the form x_1^n, so in what follows we put $T_0 = x_1^0 = 1$.

Theorem 1

The derivation operator to an order n can be written as : for all $n \geq 1$

$$T_n = \sum_{p=0}^{n-1} C_{n-1}^{p} x_{1+p} T_{n-1-p}$$

With $T_n = (gof)^{(n)}$, $x_i = f^{(i)}$ and $C_n^p = \frac{n!}{p!(n-p)!}$

Proof

for $n = 1$, $T_1 = \sum_{p=0}^{0} x_1 C_0^0 T_0 = x_1$. We assume that $T_n = \sum_{p=0}^{n-1} C_{n-1}^p x_{1+p} T_{n-1-p}$, we have :

$T_{n+1} = T_n^{'} = \sum_{p=0}^{n-1} C_{n-1}^p (x_{1+p} T_{n-1-p})^{'}$

$T_{n+1} = C_{n-1}^{n-1} x_{n+1} T_0 + \sum_{p=0}^{n-2} C_{n-1}^p x_{2+p} T_{n-1-p} + \sum_{p=1}^{n-1} C_{n-1}^p x_{1+p} T_{n-p} + C_{n-1}^0 x_1 T_n$

$T_{n+1} = C_{n-1}^{n-1} x_{n+1} T_0 + \sum_{p=1}^{n-1} (C_{n-1}^p + C_{n-1}^{p-1}) x_{1+p} T_{n-p} + C_{n-1}^0 x_1 T_n$

Indeed, since $C_{n-1}^{n-1} = C_n^n,\ T_0 = T_{n-n},\ C_{n-1}^0 = C_n^0,\ T_n = T_{n-0}$.

Consequently,

$$T_{n+1} = \sum_{p=0}^{n} C_n^p x_{1+p} T_{n-p}$$

5 New Formula to Calculate $(gof)^{(n)}$

Based on Theorem 1 we propose a new formula to calculate directly the n-th derivative of a composite function $(gof)^{(n)}$.

Theorem 2

For all $n \geq 1$,

$$(gof)^{(n)} = \sum_{p=1}^{n} (g^{(p)} of) \{ \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} f^{(1+a_i)} \}$$

With,

$$\sigma(i) = \begin{cases} 1, & si \quad i = 1 \\ i + \Sigma_{j<i} a_j, & si \quad i \geq 2 \end{cases}$$

Application Example $(gof)^{(3)}$:

Let us put : $A_p = \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} x_{1+a_i}$

$p = 1 \Rightarrow A(1) = \sum_{a_1=2} \prod_{i=1}^{1} C_{3-\sigma(a_i)}^{a_i} \prod_{i=1}^{1} x_{1+a_i} = C_2^2 x_3 = x_3$

$p = 2 \Rightarrow A(2) = \sum_{a_1+a_2=1} \prod_{i=1}^{2} C_{3-\sigma(a_i)}^{a_i} \prod_{i=1}^{2} x_{1+a_i} = 3x_1x_2$

$p = 3 \Rightarrow A(3) = \sum_{a_1+a_2+a_3=0} \prod_{i=1}^{3} C_{3-\sigma(a_i)}^{a_i} \prod_{i=1}^{3} x_{1+a_i} = x_1^3$

Because, $T_3 = A(1) + A(2) + A(3)$, we have $\Delta_3 = \sigma_1 x_3 + \sigma_2(3x_1x_2) + \sigma_3 x_1^3$.

Consequently : $(gof)^{(3)} = (g^{(3)} of)(f^{'})^3 + 3(g^{(2)} of) f^{'} f^{(2)} + (g^{'} of) f^{(3)}$

Corollary

- Let us take $g = f^{-1}$, where f is a bijective and C^n function, f^{-1} its inverse is also C^n function.

We have:

$$(gof)^{(n)} = (x)^{(n)} = 0 \quad pour \quad n \geq 2$$

On the other hand for $n \geq 2$

$$(gof)^{(n)} = \sum_{p=1}^{n} ((f^{-1})^{(p)} of)\{ \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} f^{(1+a_i)} \} = 0$$

For $f(x) = y$,f^{-1} is the solution of the equation

$$\sum_{p=1}^{n} ((f^{-1})^{(p)}(y))\{ \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} y^{(1+a_i)} \} = 0$$

- Let us take $f(x) = \ln(x)$ and $g(x) = \exp(x)$, we have :

$$(gof)^{(n)} = (x)^{(n)} = 0 \quad pour \quad n \geq 2$$

On the other hand for $n \geq 2$ and $x > 0$:

$$(gof)^{(n)} = \sum_{p=1}^{n} ((\exp(x))^{(p)} of)\{ \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} \ln(x)^{(1+a_i)} \} = 0$$

Since $\quad f^{(1+a_i)}(x) = \frac{(-1)^{a_i} a_i!}{x^{1+a_i}}$, we have :

$$\sum_{p=1}^{n} x\{ \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} \frac{(-1)^{a_i} a_i!}{x^{1+a_i}} \} = 0$$

Consequently :

$$\sum_{p=1}^{n} \{ \sum_{\sum_{i=1}^{p} a_i = n-p} \prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \prod_{i=1}^{p} (-1)^{ai} a_i! x^{-a_i} \} = 0$$

The choice of the value of x allows us to obtain some algebraic relations between these two terms :

$$\prod_{i=1}^{p} C_{n-\sigma(a_i)}^{a_i} \; and \; \prod_{i=1}^{p} (-1)^{ai} a_i!$$

6 Conclusion

In this work we have proposed a simpler formula than Faà Di Bruno. In a future work, we will use this result for the generating functions related to the probability distribution, random series which require the passage through the composition of the generating functions, moments of order k and factorial moments.

Competing Interests

Authors have declared that no competing interests exist.

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