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Total Domination Number of Generalized Petersen Graphs P(ck,k)

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Authors' contributions

This work was carried out in collaboration between all authors. Author WL designed the study and supervised the work. Authors FL and LF proved Theorem 1. Authors WC and JF proved Theorem 2. Author WL wrote the first draft of the manuscript. Authors LF and JF wrote the rest of the manuscript. All authors read and approved the final manuscript.

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Abstract

A set S of vertices of a graph $G = (V, E)$ with no isolated vertex is a total dominating set if every vertex of $V(G)$ is adjacent to some vertex in S. The total domination number is the minimum cardinality of a total dominating set of G. In this paper, we study the total domination in generalized Petersen graphs $P(ck,k)$. The upper bounds of the total domination number of generalized Petersen graphs $P(3k,k)$ and $P(4k,k)$ are obtained.

Keywords: Total domination number; generalized Petersen graph; upper bound.

1 Introduction

It is well-known that an interconnection network can be modeled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore, various problems in

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networks can be studied by graphs theoretical methods. Dominations have become one of the major areas in Graph Theory after more than 30 years' development [1]. Among the domination-type parameters that have been studied, two most fundamental ones are the domination number and the total domination number. The difference between them is whether each vertex in a dominating set can be viewed as dominated by itself. Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. The open neighborhood of v, denoted by $N(v)$, is a set $\{u \mid uv\in E\}$. And the closed neighborhood of v, denoted by $N[v]$, is a set $N(v)\cup\{v\}$ [2]. A set S of vertices of a graph G is a dominating set if every vertex in $V\backslash S$ is adjacent to a vertex in S. If S is a dominating set of G and every vertex in V is adjacent to a vertex in S, then S is called a total dominating set. A total dominating set is also a dominating set. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the total domination number of G and a $\gamma_t(G)$-set is a total dominating set of G with cardinality $\gamma_t(G)$. This concept was introduced by Cockayne et al. [3] and was studied by several authors [4-6].

The generalized Petersen graph $P(n,k)$ is the graph with vertex set $V=U\cup W$ where $U=\{u_0,u_1,\ldots,u_{n-1}\}$ and $W=\{w_0,w_1,\ldots,w_{n-1}\}$, and edge set $E=\{u_iu_{i+1},u_iw_i,w_iw_{i+k} \mid 0\le i\le n-1,$ subscripts modulo $n\}$. In recent years, there have been many results on generalized Petersen graph and related to domination parameters [7-13]. Cao et al. [14] studied the total domination of generalized Petersen graphs and obtained the exact value of the total domination number of generalized Petersen graphs $P(n,2)$. In this paper, we study the total domination in generalized Petersen graphs $P(ck,k)$ where $c=3,4$. The upper bounds are found for $P(3k,k)$ and $P(4k,k)$. The following theorem will be used in this paper.

Theorem A. [15] If G is a k-regular graph with order n, then $\gamma_t(G)\ge\lceil n/k\rceil$.

2 Total Domination Number of Generalized Petersen Graph $P(ck,k)$

Theorem 1. For generalized Petersen graphs $P(3c,3)$, we have $\gamma_t(P(3c,3))=8$ if $c=3$ and $\gamma_t(P(3c,3))\le 3c-2$ if $c>3$.

Proof. In case $c>3$, the inner circle of $P(3c,3)$ is constructed by three circles C_1,C_2,C_3 of length c and the outer circle of $P(3c,3)$ is a circle $C=u_1u_2\cdots u_{3c}u_1$ of length $3c$. C_1, C_2 and C_3 are described as $C_1=w_1w_4w_7\cdots w_{3c-2}w_1$, $C_2=w_2w_5w_8\cdots w_{3c-1}w_2$ and $C_3=w_3w_6w_9\cdots\ w_{3c}w_3$. Let $T_1=\{w_1,w_2,\ldots,w_{3c-6},$ $u_{3c-4},w_{3c-4},u_{3c-1},w_{3c-1}\}$ as illustrated in Fig. 1, then $|T_1|=3c-2$. Since (u_{3c-4},w_{3c-4}), (u_{3c-1},w_{3c-1}) and (w_iw_{i+3}) $(1\le i\le 3c-9)$ are edges in $P(3c,3)$, T_1 is total dominated by T_1. Further, $w_1\cdots w_{3c-6}$ dominates u_1, $u_2 ... u_{3c-6}$, u_{3c-4}, u_{3c-1} and u_{3c-4}, u_{3c-1} dominates u_{3c-5}, u_{3c-3}, u_{3c-2} and u_{3c} ,thus all the vertices on the outer circle are total dominated by T_1. w_1, $w_4 ... w_{3c-8}$ total dominates w_{3c-2}, w_4, w_1, $w_7 ... w_{3c-11}$, w_{3c-5} of C_1, w_2, $w_5 ... w_{3c-7}$ total dominates w_{3c-1}, w_5, w_2, $w_8 ... w_{3c-10}$, w_{3c-4} of C_2 and w_3, $w_6,\ldots,w_{3c-6}$ total dominates w_{3c}, w_6, w_3, $w_9 ... w_{3c-9}$, w_{3c-3} of C_3 .Thus T_1 is a total dominating set and $\gamma_t(P(3c,3))\le|T_1|=3c-2$.

In case c=3, the inner circle of $P(9,3)$ is constructed by three circles C_1,C_2,C_3 of length 3 and the outer circle of $P(9,3)$ is a circle $C=u_1u_2\cdots u_9u_1$ of length 9 . C_1,C_2,C_3 are described as $C_1=w_1w_4w_7w_1$, $C_2=w_2w_5w_8w_2$ and $C_3=w_3w_6w_9w_3$ .Since $P(9,3)$ is a 3-regular graph, by Theorem A, $\gamma_t(P(9,3))\ge 6$. Let $T_2=\{u_1,w_1,u_3,w_3,u_5,\ w_5,u_7,w_7\}$ as illustrated in Fig. 2, then $|T_2|=8$. Since (u_i,w_i) $(i=1,3,5,7)$ are edges of $P(9,3)$, all vertices of T_2 are total dominated by T_2. Further, u_1,u_3,u_5,u_7 total dominates the vertices u_2, u_4, u_6, u_8, u_9 of outer circle. Thus all the vertices of outer circle are total dominated by T_2.

About the vertices of inner circles, w_1 , w_7 total dominate w_4 , w_5 total dominates w_2 , w_8 , and w_3 total dominates w_6 , w_9 .Thus T_2 is a total dominating set of $P(9,3)$. Therefore, $\gamma_t(\mathrm{P}(9,3)) \le |\mathrm{T}_2|=8$. From above, $6 \le \gamma_t(\mathrm{P}(9,3)) \le 8$.

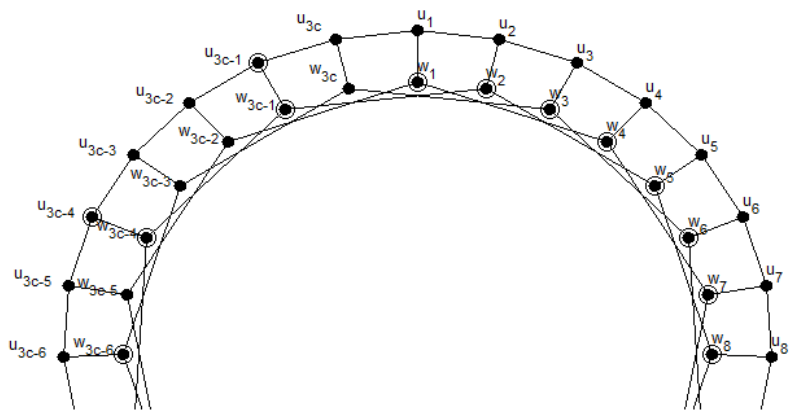

Fig. 1. The total dominating set T_1 of $P(3c,3)$ with $c>3$

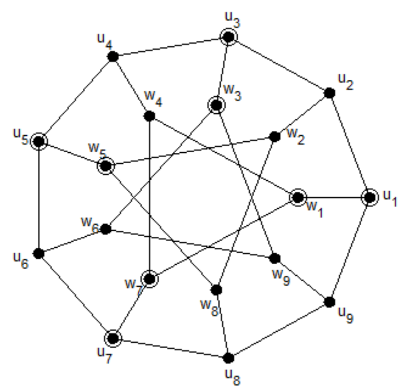

Fig. 2. The total dominating set T_2 of $P(9,3)$

Next we will show that $\gamma_t(\mathrm{P}(9,3)) \notin \{6,7\}$.

First, $\gamma_t(\mathrm{P}(9,3)) \neq 6$ .If not so, we assume that $\gamma_t(\mathrm{P}(9,3))=6$, then there exists a minimum total dominating set S with $|S|=6$. Thus, for any two vertices of S , they won't have the same neighbor. Since S doesn't have isolate vertex, at least two vertices of S are adjacent. If these two vertices belong to inner circles, they will have same neighbor, a contradiction. Thus, at least one of them belongs to outer circle. If two of them belong to outer circle, we assume these two vertices are u_1, u_2 . Then u_5 or w_3 must belong to S . If $u_5 \in S$, then the vertex which may belong to S and is adjacent to u_5 is only u_6 . While there doesn't exist any other vertex of outer circle belonging to S . But there don't exist two vertices of the rest of the vertices satisfying the condition that they belong to S and are adjacent, a contradiction. If $w_3 \in S$, then there doesn't exist any other vertex of S adjacent to w_3 , a contradiction. If one of them belongs to outer circle and the other one belongs to inner circle, we assume these two vertices are u_1, w_1 . Then u_5 or w_5 must belong to S . If $u_5 \in S$, then the vertices which may belong to S and adjacent to u_5 are w_5 and u_6. While there doesn't exist any other vertex of V belonging to S, a contradiction. By the symmetric of $P(9,3)$, we can see that other selection of vertices also contradicts the assumption.

Second, $\gamma_t(\mathrm{P}(9,3)) \neq 7$. If not so, we assume that $\gamma_t(\mathrm{P}(9,3))=7$, then there exists a minimum total dominating set S' with $|S'|=7$ and there is no isolated vertex in S'. Thus, there is a induced subgraph P_3 by three vertices in S'. We can see that for every path P_3 in $P(9,3)$, there don't exist four vertices that aren't isolate in S' and can total dominate the rest ten vertices. Therefore, $\gamma_t(\mathrm{P}(9,3)) \neq 7$.

From above, $\gamma_t(P(9,3)) = 8$.

Theorem 2. For generalized Petersen graphs $P(4c,4)$, we have $\gamma_t(P(4c,4)) = 8c/3$ if $4c \equiv 0(\bmod 3)$, $\gamma_t(P(4c,4)) \le (8c+10)/3$ if $4c \equiv 1(\bmod 3)$ and $\gamma_t(P(4c,4)) \le (8c+8)/3$ if $4c \equiv 2(\bmod 3)$.

Proof. Case 1 $4c \equiv 0(\bmod 3)$.

The inner circle of $P(4c,4)$ is constructed by four circles C_1, C_2, C_3, C_4 of length c and the outer circle of $P(4c,4)$ is a circle $C = u_1u_2\cdots u_{4c}u_1$ of length $4c$. C_1, C_2, C_3 and C_4 are described as $C_1 = w_1w_5w_9\cdots w_{4c-3}w_1$, $C_2 = w_2w_6w_{10}\cdots w_{4c-2}w_2$, $C_3 = w_3w_7w_{11}\cdots w_{4c-1}w_3$ and $C_4 = w_4w_8w_{12}\cdots w_{4c}w_4$. Let $T = \{u_1, w_1, u_4, w_4, \quad u_7, w_7 \cdots u_{4c-2}, w_{4c-2}\}$ as illustrated in Fig. 3, then $|T| = 8c/3$ .Since $(u_i, w_i)(1 \le i \le 4c\text{-}2)$ are edges in $P(4c,4)$, vertices in T are total dominated. Further, the vertices $u_1, u_4, u_7 \ldots u_{4c-2}$ total dominate u_{4c}, u_2, u_3, u_5, u_6, $u_8 \ldots u_{4c-3}$, u_{4c-1} of C. Thus, all vertices on outer circle are total dominated by T. For inner circles, w_1 $w_{13} \ldots w_{4c-11}$ total dominate w_{4c-3}, w_5, w_9, $w_{17} \ldots w_{4c-7}$, w_{4c-15} of C_1. Similarly, $w_{10}, w_{22} \ldots w_{4c-2}$ total dominate w_6, w_{14}, w_{18}, $w_{26} \ldots w_{4c-6}$, w_{4c+2} of C_2, w_7, $w_{19}, \ldots w_{4c-5}$ total dominate w_3, w_{11}, w_{15}, $w_{23} \ldots w_{4c-9}$, w_{4c-1} of C_3 and w_4, $w_{16}, \ldots w_{4c-8}$ total dominate w_{4c}, w_8, w_{12}, $w_{20} \ldots w_{4c-12}$, w_{4c-4} of C_3 .Thus, T is a total dominating set of $P(4c,4)$ and $\gamma_t(P(4c,4)) \le |T| = 8c/3$. On the other hand, since $P(4c,4)$ is a 3-regular graph, by Theorem A, we have $\gamma_t(P(4c,4)) \ge \lceil 8c/3 \rceil = 8c/3$, which implies that $\gamma_t(P(4c,4)) = 8c/3$.

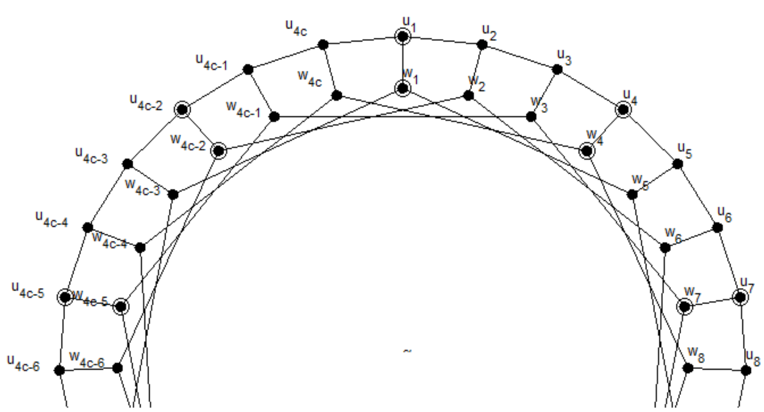

Fig. 3. The total dominating set T of $P(4c,4)$ with $4c \equiv 0(\bmod 3)$

Case 2 $4c \equiv 1(\bmod 3)$.

Let $T = \{u_1, w_1, u_4, w_4, u_7, w_7 \cdots u_{4c-3}, w_{4c-3}, w_{4c}, u_{4c}, u_{4c-1}, u_2\}$ as illustrated in Fig. 4, then T is a total dominating set of $P(4c,4)$. Therefore, $\gamma_t(P(4c,4)) \le (8c+10)/3$.

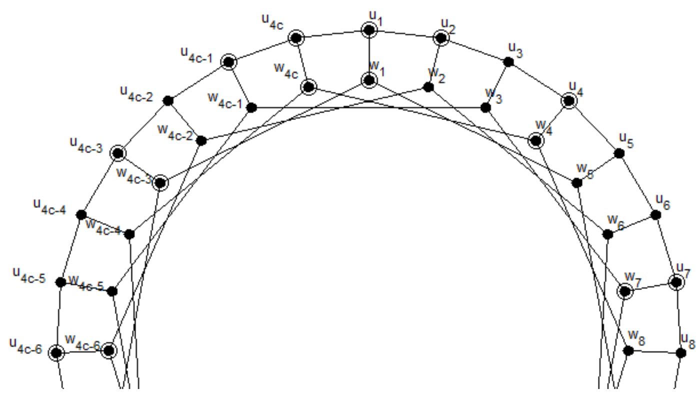

Fig. 4. The total dominating set T of $P(4c,4)$ with $4c \equiv 1(\bmod 3)$

Case 3 $4c \equiv 2(\bmod 3)$.

Let $T = \{u_1, w_1, u_4, w_4, u_7, w_7 \cdots u_{4c-4}, w_{4c-4}, u_{4c-2}, w_{4c-2}, u_{4c-1}, w_{4c-1}\}$ as illustrated in Fig. 5, then T is a total dimnating set of $P(4c,4)$. Therefore, $\gamma_t(P(4c,4)) \le (8c+8)/3$.

From above, the assertion follows.

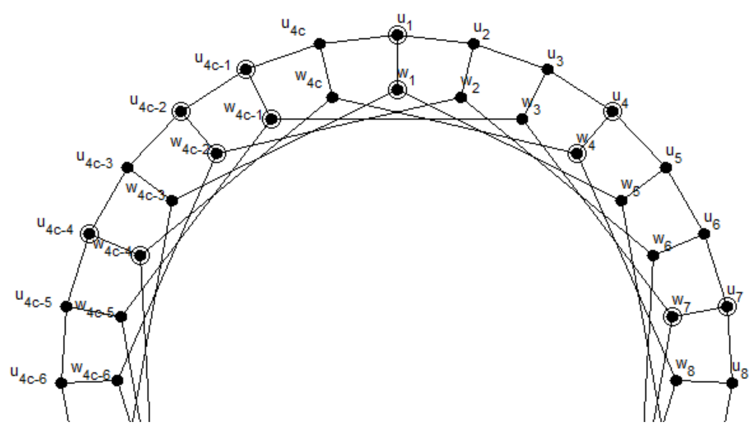

Fig. 5. The total dominating set T of $P(4c,4)$ with $4c \equiv 2(\bmod 3)$

3 Conclusion

The upper bounds of the total domination number of generalized Petersen graphs *P(3k,k)* and *P(4k,k)* are obtained. For generalized Petersen graphs $P(3c,3)$, we have $\gamma_t(P(3c,3)) = 8$ if $c = 3$ and $\gamma_t(P(3c,3)) \le 3c-2$

if $c > 3$. For generalized Petersen graphs $P(4c,4)$, we have $\gamma_t(P(4c,4)) = 8c/3$ if $4c \equiv 0(\bmod 3)$, $\gamma_t(P(4c,4)) \le (8c+10)/3$ if $4c \equiv 1(\bmod 3)$ and $\gamma_t(P(4c,4)) \le (8c+8)/3$ if $4c \equiv 2(\bmod 3)$.

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Competing Interests

Authors have declared that no competing interests exist.

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