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Asymptotic Properties of Estimators in Stochastic Differential Equations with Additive Random Effects

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

A stochastic differential equation (SDE) defined N independent stochastic processes $(X_i(t), t \in 0, T_i, i=1, \ldots, N,$ the drift term depends on the random variable ϕ_i. The distribution of the random effect ϕ_i depends on unknown parameters. When the drift term is defined linearly on the random effect ϕ_i (additive random effect) and ϕ_i has Gaussian Distribution, we propose an alternative route to prove asymptotic properties of Maximum Likelihood Estimator (MLE) by verifying the regularity conditions required through existing relevant theorems. We consider the Bayesian approach to learn the hyper parameters and proving asymptotic properties of the posterior distribution of the hyper parameters in the SDE's model.

Keywords: Asymptotic normality; consistency; maximum likelihood estimator; mixed effects stochastic differential equations; posterior normality; posterior consistency.

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1 Introduction

The use of stochastic differential equations is very essential in financial economics, biological sciences and physical sciences. The SDE model's parameters and functions estimate usually from observations of the process. Models based on random effect prefer an increasing popularity. The estimation of ML of the model parameters is intractable as the likelihood function is unclear in most cases, except in Ditlevsen et al. and Picchini et al. ([1,2]), they used a special case of a mixed-effects Brownian motion with drift to reach the likelihood function which gives explicit parameters estimators.

Many authors proposed approximations of the likelihood function, Laplace's approximation Vonesh and Wolfinger ([3,4]), approximation by Hermit polynomials Aït-Sahalia, [5] and approximation based on linearization Bealikl and Sheiner [6].

Delattre et al. [7] considered a special case by multiplying the drift by the random effect, where $b(x, \phi_i)$ is linear in ϕ_i $(b(x, \phi_i) = \phi_i\, b(x))$. The consistency and the asymptotic normality of the (MLE) proved for Gaussian distribution random effect.

Maitra et al. [8], used the alternative route to prove the consistency and asymptotic normality in the SDE of MLE. The asymptotic properties of the posterior distribution (independent identical and independent non-identical) proved for linear drift [9]. Where the study conducted for the model proposed in [7].

Alsukaini et al. [10], considered nonlinearity in the diffusion term of the SDE where $\sigma(x, \phi_i) = \sigma(x)/\phi_i$ with ϕ_i has exponential and Gaussian distribution respectively. The study concluded proves the consistency and the asymptotic normality of the (MLE).

Delattre, et al. [7] studied the stochastic differential equations (SDE's) of the form:

$$dX_i(t) = b(X_i(t), \phi_i)dt + \sigma\big(X_i(t)\big)dW_i(t), with \;\; X_i(0) = x^i, \; i = 1, \dots, N \tag{1}$$

Here, the stochastic process ($X_i(t), t \geq 0, i = 1, \dots, N$) is assumed to be continuously observed on the time interval $[0, T_i]$ with $T_i > 0$, and $(x^i, \; i = 1, \dots, N)$ are the initial values of the ith process. Where the processes $(W_1, \dots, W_N)$ are independent standard Brownian motions, $(\phi_1, \dots, \phi_N)$ are independently and identically distributed $(i.i.d)$ random variables with common distribution $g(\varphi, \theta)d\nu(\varphi)$ for all θ, $g(\varphi, \theta)$ is a density with respect to a dominating measure on $\mathbb{R}^m$, where $\mathbb{R}$ is the real line and m is the dimension. Also the processes $(W_1, \dots, W_N)$ are independent of random variables $(\phi_1, \dots, \phi_N)$. Here θ known parameter belonging to a set $\Theta \subset \mathbb{R}^m$ which be estimated. The drift function $b(x; \varphi)$ is a known function defined on $\mathbb{R} \times \mathbb{R}^m$ and real-valued. The diffusion coefficient the likelihood $\sigma: \mathbb{R} \to \mathbb{R}$ is a known real-valued function.

In Delattre et al. [7] a regularity conditions proposed to solve equation (1) and the likelihood obtained as follows:

$$\lambda_i(X_i, \varphi) = \int_{\mathbb{R}^d} L_i(X_i, \varphi) g(\varphi, \theta) d\nu(\varphi) \;, \qquad i = 1, \dots, N,$$

where

$$L_i(X_i, \varphi) = exp\left(\int_0^{T_i} \frac{b(X_i(t), \phi_i)}{\sigma^2(X_i(t))}\, dX_i(s) - \frac{1}{2}\int_0^{T_i} \frac{b^2(X_i(t), \phi_i)}{\sigma^2(X_i(t))}\, ds\right), \qquad i = 1, \dots, N. \tag{2}$$

Where the likelihood depending upon θ, admits a relatively simple form composed of the following sufficient statistics:

$$U_i = \int_0^{T_i} \frac{b(X_i(t))}{\sigma^2(X_i(t))}\, dX_i(s) \;, \qquad V_i = \int_0^{T_i} \frac{b^2(X_i(t))}{\sigma^2(X_i(t))}\, ds \;, \quad i = 1, \dots, N \tag{3}$$

And assume that $V_i < \infty$ almost surely for every $i \geq 1$.

Alkreemawi et al. [11] considered the addition case in the drift term where $b(x, \phi_i)$ is linear in ϕ_i($b(x, \phi_i) = \phi_i + b(x)$). The sufficient statistics (3) adopted for the considered model and a new sufficient statistics shown below investigated,

$$Y_i = \int_0^T \frac{1}{\sigma^2(X_i(t))}\, dX_i(s)\ , \quad Z_i = \int_0^T \frac{1}{\sigma^2(X_i(t))}\, ds\ , \quad D_i = \int_0^T \frac{b(X_i(t))}{\sigma^2(X_i(t))}\, ds\ , \tag{4}$$

the exact likelihood is given by

$$\Lambda_N(\theta) = \prod_{i=1}^{N} \lambda_i(X_i, \theta)$$

Where

$$\lambda_i(X_i, \theta) = \int_{\mathbb{R}^d} g(\varphi, \theta) exp\left(\varphi(Y_i - D_i) - \frac{\varphi^2}{2} Z_i + \left(U_i - \frac{1}{2} V_i\right)\right) dv(\varphi) \tag{5}$$

Assuming that $(\varphi, \theta) \equiv N(\mu, w^2)$, the following forms of $\lambda_i(X_i, \theta)$ obtained:

$$\lambda_i(X_i, \theta\ ) = \frac{1}{\sqrt{1+w^2 Z_i}} exp\left[-\frac{Z_i}{1+w^2 Z_i}\left(\mu - \frac{(Y_i - D_i)}{Z_i}\right)^2\right] exp\left[\frac{(Y_i - D_i)^2}{2Z_i}\right] \times exp\left[U_i - \frac{1}{2} V_i\right], \tag{6}$$

where $\theta = (\mu, w^2) \in \mathbb{R} \times \mathbb{R}^+$ and studied asymptotic properties of ϕ_i when ϕ_i has a Gaussian distribution.

In this article, as an alternative, we will prove consistency and asymptotic normality of the MLE in Alkreemawi et al. [11] situation where drift $b(x, \phi_i)$ is linear in ϕ_i($b(x, \phi_i) = \phi_i + b(x)$) by verifying the regularity conditions of relevant theorems already existing in the literature. The Bayesian approach to learning the hyper parameters will consider, and proves consistency and asymptotic normality of the posterior distribution of the hyper parameters.

This paper is organized as follows, in section 2 we investigate consistency and asymptotic normality of the MLE in the SDE. In section 3 we consider the Bayesian framework, for the SDE, and prove consistency and asymptotic normality of the Bayesian posterior distribution of $\theta = (\mu, w^2)$.

2 Asymptotic Properties of MLE

2.1 Strong consistency of MLE

We used the assumptions referred as (H1), (H2) and (H3) in [7] and the assumption (H4) of Alkreemawi et al. [11] is assumed as well. The function $b(\,.\,)/\sigma(\,.\,)$ assumed not constant, and (D_1, Y_1, Z_1) has a density $f_1(d, y, z)$ with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{R}^+$ which is jointly continuous and positive on an open ball of $\mathbb{R} \times \mathbb{R}^+$. In addition to previous assumptions we propose the following assumptions to prove the consistency and asymptotic normality of estimators of θ:

(H5) i-$b(\,.\,)$ and $\sigma(x)$are C^1on $\mathbb{R}$ satisfying $b^2(x) \leq K(1 + x^2)$ and $\sigma^2(x) \leq K(1 + x^2)$ for all $x \in \mathbb{R}$, for some $K > 0$.

ii- Almost surely for each $i \geq 1$,

$$\int_0^{T_i} \frac{b^2(X_i(s))}{\sigma^2(X_i(s))} ds < \infty$$

(H6) The parameter set Θ is a compact subset of $\mathbb{R} \times \mathbb{R}^+$.
(H7) The true value θ_0 belongs to Θ^o.
(H8) The matrix $I(\theta_0)$ is invertible (see Alkreemawi et al. [11] section 3.2).

Now, we investigate consistency of the MLE by validation of regularity conditions of theorem (Theorems 7.49 and 7.54 of Schervish [12]).

Theorem 1 [12]. Let $\{x_n\}_{n=1}^{\infty}$ be conditionally $i.i.d$ given θ with density $f_1(x|\theta)$ with respect to a measure v on a space $(\chi^1, \mathcal{B}^1)$. Fix $\theta_0 \in \Theta$, and define, for each $M \subseteq \Theta$ and $x \in \chi^1$.

$$Z(M, x) = \inf_{\gamma \in M} log \frac{f_1(x|\theta_0)}{f_1(x|\gamma)}$$

Assume that for each $\theta \neq \theta_0$, there is an open set N_θ such that $\theta \in N_\theta$ and that $E_{\theta_0} Z(N_\theta, X_i) > -\infty$. Also assume that $f_1(x|.)$ is continuous at θ for every θ, a.s. $[P_{\theta_0}]$. Then if $\hat{\theta}_n$ is the MLE of θ corresponding to n observations, It holds that $lim_{n\to\infty} \hat{\theta}_n = \theta_0$ a.s. $[P_{\theta_0}]$.

Proposition 2.1.1. Under conditions (H5) and (H8). Then the MLE is strongly consistent, in other words $lim_{n\to\infty} \hat{\theta}_n = \theta_0$ a.s. $[P_{\theta_0}]$.

Proof. For verifying the conditions of Theorem 1 in our situation, for any x we note that $f_1(x|\theta) = \lambda_1(x, \theta) = \lambda(x, \theta)$ continuous in θ, which is given by (5), and explicit form by (6). Hence, for each $\theta \neq \theta_0$ we obtain.

$$log \frac{f_1(x|\theta_0)}{f_1(x|\gamma)} = \frac{1}{2} log \left(\frac{1 + w^2 Z_1}{1 + w_0^2 Z_1} \right) + \frac{1}{2} \frac{(w_0^2 - w^2)(Y_1 - D_1)^2}{(1 + w^2 Z_1)(1 + w_0^2 Z_1)} + \frac{\mu^2 Z_1}{2(1 + w^2 Z_1)} - \frac{\mu(Y_1 - D_1)}{(1 + w^2 Z_1)} - \frac{\mu_0^2 Z_1}{2(1 + w_0^2 Z_1)} + \frac{\mu_0(Y_1 - D_1)}{(1 + w_0^2 Z_1)}$$

$$\geq -\frac{1}{2} \left\{ log \left(1 + \frac{w^2}{w_0^2} \right) + \frac{|w^2 - w_0^2|}{w^2} \right\} - \frac{1}{2} |w_0^2 - w^2| \left(\frac{Y_1 - D_1}{1 + w_0^2 Z_1} \right)^2 \left(1 + \frac{w^2}{w_0^2} \right) - |\mu| \left| \frac{Y_1 - D_1}{1 + w_0^2 Z_1} \right| \left(1 + \frac{|w_0^2 - w^2|}{w^2} \right) - \left| \frac{\mu_0^2 Z_1}{2(1 + w_0^2 Z_1)} \right| - \left| \frac{\mu_0(Y_1 - D_1)}{(1 + w_0^2 Z_1)} \right|$$

From Lemma 3.1.1 in Alkreemawi et al. [11], we noted that $E_{\theta_0} \left(\frac{Y_1 - D_1}{1 + w_0^2 Z_1} \right)^2, E_{\theta_0} \left| \frac{Y_i - D_i}{1 + w_0^2 Z_i} \right|$ and $E_{\theta_0} \left(\frac{\mu_0^2 Z_1}{2(1 + w_0^2 Z_1)} \right)$ are finite when we consider $N_\theta = (\underline{\mu}, \bar{\mu}) \times (\underline{w}^2, \bar{w}^2)$, it follows that $E_{\theta_0} Z(N_\theta, X_i) > -\infty$. Thus, $lim_{n\to\infty} \hat{\theta}_n = \theta_0$ a.s. $[P_{\theta_0}]$.

2.2 Asymptotic normality of MLE

The theorem introduced by Schervish [12] (Theorem 7.63) used to investigate asymptotic normality of MLE.

Theorem 2 [12]. Let Θ be a subset of $\mathbb{R}^m$, and let $\{x_n\}_{n=1}^{\infty}$ be conditionally $i.i.d$ given θ with density $f_1(.|\theta)$. Let $\hat{\theta}_n$ be an MLE. Assume that $\hat{\theta}_n \xrightarrow{P} \theta_0$ under P_θ for all θ. Assume that $f_1(x|\theta)$ has continuous second partial derivatives with respect to θ and that differentiation can be passed under the integral sign. Assume that there exists $H_r(x, \theta)$ such that, for each $\theta_0 \in int(\Theta)$ and each k, j,

$$sup_{\|\theta-\theta_0\|\le r}\left|\frac{\partial^2}{\partial\theta_k\partial\theta_j}logf_{X_1|\psi}(x|\theta_0)-\frac{\partial^2}{\partial\theta_k\partial\theta_j}logf_{X_1|\psi}(x|\theta)\right|\le H_r(x,\theta_0), \tag{7}$$

with

$$lim_{r\to 0}\, E_{\theta_0}H_r(x,\theta_0)=0 \tag{8}$$

Assume that the Fisher information matrix $I(\theta)$ is finite and non-singular. Then, under P_{θ_0},

$$\sqrt{n}\big(\hat{\theta}_n-\theta_0\big)\overset{\mathcal{L}}{\to}N\big(0,I^{-1}(\theta_0)\big) \tag{9}$$

Proposition 2.2.1. Under conditions (H5) and (H6). Then the MLE is asymptotically normally distributed as (9).

Proof. The proof of Proposition 3.1.2. [11], showed that the differentiation was passed under the integral sign, and in proof of Proposition 3.3.1 in Alkreemawi et al. [11], we proved almost sure consistency of the MLE $\hat{\theta}_n$, hence $\hat{\theta}_n\overset{P}{\to}\theta_0$ under P_θ for all θ. From proof of Proposition 3.2.1 in Alkreemawi et al. [11], we note that,

$$\frac{\partial^2}{\partial\mu^2}logf_1(x|\theta)=-\xi_1(w^2),\qquad \frac{\partial^2}{\partial\mu\partial w^2}logf_1(x|\theta)=-\eta_1(\theta)\xi_1(w^2) \tag{10}$$

$$\frac{\partial^2}{\partial w^2\partial w^2}logf_1(x|\theta)=-\frac{1}{2}\big(2\eta_1^2(\theta)\xi_1(w^2)-\xi_1^2(w^2)\big). \tag{11}$$

Where $\eta_i(\theta)=\left(\frac{(Y_i-D_i)-\mu Z_i}{1+w^2Z_i}\right)$ and $\xi_i(w^2)=\frac{Z_i}{1+w^2Z_i}$. The derivatives (10) and (11) in our case $\frac{\partial^2}{\partial\theta_k\partial\theta_j}logf_1(x|\theta)$ are differentiable in $\theta=(\mu,w^2)$. To verify Two conditions as mentioned in Theorem 2, (see (7) and (8)), we should prove the derivatives (10) and (11) having a finite expectation under E_{θ_0} ($\theta_0=(\mu_0,w_0^2)$). We have the upper bound:

$$0<\frac{1+w^2Z_1}{1+w_0^2Z_1}<1+\frac{w^2}{w_0^2}$$

Introducing the function $h(x)=x-1-logx$, which is defined on $\mathbb{R}^+$ and non-negative, we have the lower bound:

$$log\left(\frac{1+w^2Z_1}{1+w_0^2Z_1}\right)=h\left(\frac{1+w^2Z_1}{1+w_0^2Z_1}\right)+(w_0^2-w^2)\frac{Z_1}{1+w^2Z_1}\ge(w^2-w_0^2)\frac{Z_1}{1+w^2Z_1},$$

thus

$$log\left(\frac{1+w^2Z_1}{1+w_0^2Z_1}\right)\le log\left(1+\frac{w^2}{w_0^2}\right)+\frac{|w^2-w_0^2|}{w^2} \tag{12}$$

For the second term, we write:

$$0<\frac{(Y_1-D_1)^2}{(1+w^2Z_1)(1+w_0^2Z_1)}=\left(\frac{Y_1-D_1}{1+w_0^2Z_1}\right)^2\frac{1+w_0^2Z_1}{1+w^2Z_1}\le\left(\frac{Y_1-D_1}{1+w_0^2Z_1}\right)^2\left(1+\frac{w^2}{w_0^2}\right) \tag{13}$$

By Lemma 3.1.1 in Alkreemawi et al. [11] the right-hand side has finite E_{θ_0}-expectation. For the last terms, we only need to check that term $|(Y_1-D_1)/(1+w^2Z_1)|$ has finite expectation under E_{θ_0}. For this, we remark that

$$\frac{Y_1-D_1}{1+w^2Z_1}=\frac{Y_1-D_1}{1+w_0^2Z_1}\left(1+(w_0^2-w^2)\frac{Z_1}{1+w^2Z_1}\right),$$
$$\left|\frac{Y_1-D_1}{1+w^2Z_1}\right|\leq\left|\frac{Y_1-D_1}{1+w_0^2Z_1}\right|\left(1+\frac{|w_0^2-w^2|}{w^2}\right) \quad (14)$$

Which has finite E_{θ_0}-expectation by Lemma 3.1.1 in Alkreemawi et al. [11].

3 Asymptotic Properties of Bayesian Posterior

3.1 Consistency of the Bayesian posterior distribution

To prove posterior consistency in our situation, we use the following theorem (theorem 7.80 in Schervish [12]) to verify sufficient conditions that ensure posterior consistency.

Theorem 3 [12]. Let $\{x_n\}_{n=1}^{\infty}$, be conditionally iid given θ with density $f_1(x|\theta)$ with respect to a measure v on a space $(\chi^1,\mathcal{B}^1)$. Fix $\theta_0\in\Theta$, and define, for each $M\subseteq\Theta$ and $x\in\chi^1$.

$$Z(M,x)=\inf_{\gamma\in M} log\frac{f_1(x|\theta_0)}{f_1(x|\gamma)}$$

Assume that for each $\theta\neq\theta_0$, there is an open set N_θ such that $\theta\in N_\theta$ and that $E_{\theta_0}Z(N_\theta,X_i)>-\infty$. Also assume that $f_1(x|.)$ is continuous at θ for every θ, a.s. $[P_{\theta_0}]$. For $\varepsilon>0$, define $C_\varepsilon=\{\theta:\ K_1(\theta_0,\theta)<\varepsilon\}$, where

$$K_1(\theta_0,\theta)=E_{\theta_0}\left(log\frac{f_1(x|\theta_0)}{f_1(x|\theta)}\right) \quad (15)$$

Is the Kullback-Leibler divergence measure associated with observation X_1. Let π be a prior distribution such that $\pi(C_\varepsilon)>0$, for every $\varepsilon>0$. Then, for every $\varepsilon>0$ and open set N_0 containing C_ε, the posterior satisfies

$$lim_{n\to\infty}\pi(N_0|X_1,\ldots,X_n)=1,\qquad a.s.\quad [P_{\theta_0}]. \quad (16)$$

Proposition 3.1.1. Under conditions (H5) and (H8) the posterior consistency is holding, in other words $\lim_{n\to\infty}\pi(N_0|X_1,\ldots,X_n)=1,$ a.s. $[P_{\theta_0}]$.

Proof. From above, the conditions of Theorem 3 are verified in Theorem 1 in section 2.1. To complete the proof we need to ensure that a prior π exists which for every $\varepsilon>0$ gives positive probability to C_ε. Since for any $\varepsilon>0$, $K_1(\theta_0,\theta)=0$ if and only if $\theta_0=\theta$, the set C_ε is nonempty provided that $\Theta\backslash\{\theta_0\}$ is non-empty. Let $(d\pi/dv)=h$ almost everywhere on Θ, where $h(\theta)$ is any positive and continuous density on Θ with respect to the Lebesgue measure v. Now we show that $K_1(\theta_0,\theta)$ is continuous in θ. Since

$$K_1(\theta_0,\theta)=E_{\theta_0}\big(\mathcal{L}_1(\theta_0)-\mathcal{L}_1(\theta)\big),$$

Where

$$\mathcal{L}_1(\theta)=log\lambda_1(X_1,\theta)=logf_1(x|\theta)$$

Rearranging terms, we obtain:

$$\mathcal{L}_1(\theta_0) - \mathcal{L}_1(\theta) = \frac{1}{2}\log\left(\frac{1+w^2Z_1}{1+w_0^2Z_1}\right) + \frac{1}{2}\frac{(w_0^2-w^2)(Y_1-D_1)^2}{(1+w^2Z_1)(1+w_0^2Z_1)} + \frac{\mu^2Z_1}{2(1+w^2Z_1)}$$

$$-\frac{\mu(Y_1-D_1)}{(1+w^2Z_1)} - \frac{\mu_0^2Z_1}{2(1+w_0^2Z_1)} + \frac{\mu_0(Y_1-D_1)}{(1+w_0^2Z_1)}$$

The function $\theta \to \mathcal{L}_1(\theta_0) - \mathcal{L}_1(\theta)$ is continuous. For all $\theta = (\mu, w^2) \in \left|\underline{\mu}, \bar{\mu}\right| \times \left|\underline{w}^2,\ \bar{w}^2\right| \subset \mathbb{R} \times \mathbb{R}^+$, from inequalities (12)-(13)-(14), we can easily get an upper bound for $|\mathcal{L}_1(\theta_0) - \mathcal{L}_1(\theta)|$ which has finite E_{θ_0}-expectation and is uniform on the interval $\left[\underline{\mu}, \bar{\mu}\right] \times \left[\underline{w}^2,\ \bar{w}^2\right]$. Because of the continuity of the Kullback information and the compact parameter space Θ, therefore $K_1(\theta_0, \theta)$ is uniformly continuous on Θ. Hence, for any $\varepsilon > 0$, there is δ_ε independent of θ, such that $\|\theta - \theta_0\| \le \delta_\varepsilon$ implies $K_1(\theta_0, \theta) < \varepsilon$.
Therefore,

$$\pi(C_\varepsilon) \ge \pi(\{\theta: \|\theta - \theta_0\| \le \delta_\varepsilon\}) \ge \left[\inf_{\{\theta:\, \|\theta-\theta_0\| \le \delta_\varepsilon\}} h(\theta)\right] \times v(\{\theta: \|\theta - \theta_0\| \le \delta_\varepsilon\}) > 0 \qquad (17)$$

Hence, (16) hold in our situation with any prior with continuous density w.r.t. the Lebesgue measure.

3.2 Asymptotic normality of the Bayesian posterior distribution

We combine the Theorem 7.102 and 7.89 provided in Schervish [12], to investigate asymptotic normality of posterior distribution in our situation. The above two theorems used for several regularity conditions. For our situation, we require only the first four conditions. We state the requisite conditions as below.

3.2.1 Regularity conditions:

(1) The parameter space is $\Theta \subseteq \mathbb{R}^m$ for some finite m.
(2) θ_0 is a point interior to Θ.
(3) The prior distribution of ψ has a density w. r. t. Lebesgue measure that is positive and continuous at θ_0.
(4) There exists a neighborhood $N_0 \subseteq \Theta$ of θ_0 on which $\ell_n(\theta) = log\, f(X_1, \ldots, X_n|\theta)$ is twice continuously differentiable w. r. t. all co-ordinates of θ, a.s. $[P\theta 0\,]$.

Theorem 4 [12]. Let $\{x_n\}_{n=1}^{\infty}$ be conditionally $i.i.d$ given θ. Assume the above four regularity conditions. Also assume that there exists $Hr(x, \theta)$ such that, for each $\theta_0 \in int(\Theta)$ and each k, j,

$$sup_{\|\theta-\theta_0\| \le r} \left|\frac{\partial^2}{\partial\theta_k\partial\theta_j} logf_{X_1|\psi}(x|\theta_0) - \frac{\partial^2}{\partial\theta_k\partial\theta_j} logf_{X_1|\psi}(x|\theta)\right| \le H_r(x, \theta_0), \qquad (18)$$

with

$$\lim_{r\to 0} E_{\theta_0} H_r(x, \theta_0) = 0$$

Further suppose that the conditions of Theorem 3 hold, and that the Fisher's information matrix $I(\theta_0)$ is positive definite. Now let

$$\Sigma_n^{-1} = \begin{cases} -\ell_n''(\hat{\theta}_n) & \text{if the inverse and } \hat{\theta}_n \text{ exist} \\ \mathbb{I}_m & \text{if not,} \end{cases} \qquad (19)$$

where, for any t,

$$\ell_n''(t) = \left(\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ell_N(\theta)\Big|_{\theta=t}\right)\right), \qquad (20)$$

and $\mathbb{I}_m$ is the identity matrix of order m. Thus, $\sum_n^{-1}$ is the observed Fisher's information matrix. Letting $\Psi_n = \sum_n^{-1/2}(\theta - \hat{\theta}_n)$, it follows that for each compact subset B of $\mathbb{R}^m$ and each $\varepsilon > 0$, it holds that

$$lim_{n\to\infty} P_{\theta_0}\left(sup_{\psi\in B}\left|\pi(\psi|X_1, \dots, X_n) - \tilde{\phi}(\psi)\right| > \varepsilon\right) = 0, \tag{21}$$

where $\tilde{\phi}(\,.\,)$ denotes the density of the standard normal distribution.

Proposition 3.2.1. In the assumptions provided, an asymptotic normality of posterior distributions holds.

Proof. The first three regularity conditions in Section 3.2.1 trivially hold. The remaining conditions of Theorem 4 are verified in the context of Theorem 2 in Section 2.2 Hence, equation (21) holds in our situation.

4 Conclusion

In stochastic differential equations based random effects model framework. Alkreemawi et al. [11] considered the addition case in the drift where $b(x, \phi_i)$ is linear in $\phi_i(b(x, \phi_i) = \phi_i + b(x))$, where ϕ_i has Gaussian distribution with mean μ and variance w^2, and for the likelihood of the above parameters obtained a closed form expression. They proved convergence in probability and asymptotic normality of the MLE of the parameters. In this paper, we proved strong consistency rather than weak consistency, and asymptotic normality of the MLE under weaker assumptions. We have investigated Bayesian posterior consistency in the context of Stochastic Differential Equation's (SDE's) consisting of drift functions depending linearly upon random effect parameters. In particular, we have proved posterior asymptotic properties.

Competing Interests

Authors have declared that no competing interests exist.

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