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Improve the Mean Square Solution of Second-order Random Differential Equations by Using Homotopy Analysis Method

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Authors' contributions

This work was carried out in collaboration between all authors. Authors ARK, SAMH, SLK designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Authors ARK, SAMH, SLK managed the analyses of the study andliterature searches. All authors read and approved the final manuscript.

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Original Research Article

Abstract

The Homotopy analysis method is implemented by Golmankhanehi et al. to find the expectation and variance of the approximate solutions of the second-order random differential equations [1]. In this note, we reused the Homotopy analysis method to solve the same problem and we draw some very important improvements and comments on the paper [1]. The results in this paper are coinciding with the results in [2, 3, 4, 5].

Keywords: Homotopy analysis method; random variable; variance of approximate solution; expectation of approximate solution.

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1 Introduction

The papers by Golmankhanehi et al. [1], present the Homotopy analysis method for finding the expectation and variance of the approximate solutions of the second-order random differential equations. These authors work on computation of the main statistical functions of the mean square solution, where some basic properties of variance are applied in equation (10) in [1].

The present authors are not aware of the following properties, which is used in equation (10) in [1],

$$V(\sum_{i=0}^{n} X_i) \neq \sum_{i=0}^{n} V(X_i).$$

Although, It is well known that, in probability theory and statistic [6], the variance of a random variable must be nonnegative since the variance is given by [1].

$Var(X) = E((X-\mu)^2)$, where $E(X) = \mu$

Golmankhanehi et al. [1] discussed six illustrative examples in section 3. The numerical results are presented graphically in Figs. 1-12. We observed that the numerical results of the variance approximation solution by using HAM in [1] are ambiguous since the variance curve is negative as shown in Figs. 4, 6, and 12.

It is worth noting that from equation (2) in [1], the homotopy analysis method coincides with the homotopy Perturbation method when $\hbar = -1$. As a result the expectation and the variance of the approximate solutions must be consistent with that found by Khalaf [2].

The goal of this work is to improve equation (10) in [1]. A variety of examples using the correct expression will be presented. In order to achieve this we let $x_N(t) = X_0(t) + \sum_{n=1}^{N} X_n(t)$ (equation (7) in [1]), be an approximate solution of equation (2) by using the homotopy analysis method . Then we utilize the independence between Y_0, Y_1 and A, to yield

$$E[x_N(t)] = E[X_0(t)] + \sum_{n=1}^{N} E[X_n(t)],$$

$$V[x_N(t)] = \sum_{j=0}^{N}\sum_{i=0}^{N} \mathrm{Cov}(X_i(t), X_j(t)),$$

where

$$\mathrm{Cov}(X_i(t), X_j(t)) = E(X_i(t)X_j(t)) - E(X_i(t))E(X_j(t)), \quad i, j = 0, 1, 2, ..., n.$$

2 Homotopy Analysis Method

The homotopy analysis method (HAM) has been introduced by Liao [7]. This method is a powerful and useful semi-numerical method for finding approximate solutions of linear and nonlinear differential equations. We observe that HAM is very prevalent in the current literature, see e.g. the accounts in the books by Liao [8, 9], and the recent papers by Dehghan and Salehi [10], Hassan and Mehdi [11], Das et al. [12], and references therein. To clarify the basic ideas of HAM, we consider the following differential equation

$$L[X(t)] + N[X(t), A] = g(t), \tag{2.1}$$

with initial conditions

$$X(0) = Y_0, \quad \frac{dX(t)}{dt}\mid_{t=0} = Y_1,, \tag{2.2}$$

where $L[X(t)] = d^2X(t)/dt^2$, $N[X(t), A]$ is a nonlinear operator and $g(t)$ is the source inhomogeneous term, and A, Y_0, and Y_1 are the random variables.

We now construct the zero-order deformation equation,

$$(1-q)\{L[X(t)] - L[x_0(t)]\} = q\hbar\{L[X(t)] + N[X(t), A] - g(t)\}, \tag{2.3}$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar$ a nonzero auxiliary parameter, L an auxiliary linear operator, and $x_0(t)$ is an initial approximation of the equation (2.1).

Expanding $X(t)$ in Taylor series with respect to q, we have

$$X(t) = X_0(t) + \sum_{n=1}^{\infty} X_n(t)q^n, \tag{2.4}$$

where

$$X_n(t) = \frac{1}{n!} \left.\frac{\partial^n X(t)}{\partial q^n}\right|_{q=0}. \tag{2.5}$$

The convergence of the equation (2.4) depends on the auxiliary operator $\hbar$. If it is convergent at $q = 1$, one have

$$X(t) = X_0(t) + \sum_{n=1}^{\infty} X_n(t). \tag{2.6}$$

Throughout the paper, we deal with the triplet Probabilistic space $(\Omega, \mathcal{F}, \mathcal{P})$. Thus, suppose $L_2 = L_2(\Omega, \mathcal{F}, \mathcal{P})$ is the set of second order random variables. Then the random variable $X : \Omega \to \mathbb{R} \in L_2$, if $\mathrm{E}[X^2] < +\infty$, where $\mathrm{E}[\cdot]$ is an expectation operator.

The norm on L_2 is denoted by $\|\cdot\|_2$. For example, for the random variable X we define

$\|X\|_2 = (\mathrm{E}[\mathrm{X}^2])^{\frac{1}{2}}$, in such way that $(\mathrm{L}_2, \|X\|_2)$ is a Banach space.

In addition, let T (real interval) represent the space of times, we say that $\{X(t),\ t \in T\}$ is a second order stochastic process, if the random variable $X(t) \in L_2$ for each $t \in T$.

A sequence of second order random variables $\{X_n : n \geq 0\}$ converges to $X \in L_2$, if

$$\lim_{n\to\infty} \|X_n - X\|_2 = \lim_{n\to\infty} \left(E\left[X_n - X^2\right]\right)^{\frac{1}{2}} = 0$$

To proceed from (2.6), we truncate the expansion of $X_n(t)$ at the $n = Nth$ term as follows

$$X(t) = X_0(t) + \sum_{n=1}^{N} X_n(t). \tag{2.7}$$

By using the independence between Y_0, Y_1, and A, we have

$$E[x_N(t)] = E[X_0(t)] + \sum_{n=1}^{N} E[X_n(t)], \tag{2.8}$$

$$V[x_N(t)] = \sum_{j=0}^{N}\sum_{i=0}^{N} \mathrm{Cov}(X_i(t), X_j(t)), \tag{2.9}$$

where

$$\mathrm{Cov}(X_i(t), X_j(t)) = E(X_i(t)X_j(t)) - E(X_i(t))E(X_j(t)),\ \ i, j = 0, 1, 2, ..., n.$$

The following Lemma guarantee the convergent of the sequence$E[x_N(t)]$ to $E[X(t)]$ and the sequence $V[x_N(t)]$ to $V[X(t)]$ if the sequence $x_N(t)$ converges to $X(t)$.

Lemma: [6], Let $\{X_N\}$ and $\{Y_N\}$ be two sequences of second order random variables X and Y respectively, this means,

$\lim_{N\to\infty} X_N = X$ and $\lim_{N\to\infty} Y_N = Y$ then $\lim_{N\to\infty} E[X_N Y_N] = E[XY]$. If $X_N = Y_N$ then $\lim_{N\to\infty} E[X_N^2] = E[X^2]$, $\lim_{N\to\infty} E[X_N] = E[X]$ and $\lim_{N\to\infty} V[X_N] = V[X]$.

3 Test Examples

In this section, we adopt several examples which is considered in [1, 2, 3, 4, 5] to illustrate the using of homotopy analysis method for approximating the expectation and the variance. The results coincide with results in [2, 3, 4, 5].

3.1 Example 1

Consider the random initial value problem

$$\frac{d^2X(t)}{dt^2} + A^2X(t) = 0, \quad X(0) = Y_0, \quad \frac{dX(t)}{dt}\,|_{t=0} = Y_1, \tag{3.1}$$

where,

A^2 is a Beta random variable, $A^2 \sim Be(\alpha = 2, \beta = 1)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 3$ and $E[Y_0Y_1] = 0$.

To obtain the expectation and the variance of the approximate solutions of equation (3.1), we Construct the following zero-order deformation

$$(1-q)[\frac{d^2X(t)}{dt^2} - \frac{d^2X_0(t)}{dt^2}] = q\hbar[\frac{d^2X(t)}{dt^2} + A^2\,X(t)]. \tag{3.2}$$

Next, Let

$$X(t) = X_0(t) + \sum_{k=1}^{\infty} X_k(t)\,q^k. \tag{3.3}$$

By substituting into equation (3.2), one can have

$$(1-q)[\sum_{k=1}^{\infty} \frac{d^2X_k(t)}{dt^2}\,q^k] = q\hbar[\frac{d^2X_0(t)}{dt^2} + \sum_{k=1}^{\infty} \frac{d^2X_k(t)}{dt^2}\,q^k + A^2\,X_0(t) + \sum_{k=1}^{\infty} A^2\,X_k(t)\,q^k],$$

which leads to

$$(\sum_{k=1}^{\infty} \frac{d^2X_k(t)}{dt^2}\,q^k - (1+\hbar)\sum_{k=1}^{\infty} \frac{d^2X_k(t)}{dt^2}\,q^{k+1}) = q\hbar[\frac{d^2X_0(t)}{dt^2} + A^2\,X_0(t)] + \hbar\sum_{k=1}^{\infty} A^2\,X_k(t)\,q^{k+1}.$$

Upon equating the corresponding coefficients of q^k, we have

$$\frac{d^2X_1(t)}{dt^2} = (1+\hbar)\frac{d^2X_0(t)}{dt^2} + \hbar\frac{d^2X_0(t)}{dt^2} + (1+\hbar)A^2\,X_0(t),$$

and

$$\frac{d^2X_k(t)}{dt^2} - (1+\hbar)\frac{d^2X_{k-1}(t)}{dt^2} = \hbar A^2\,X_{k-1}(t), \qquad k = 2, 3, \ldots$$

Choosing $X_0(t) = Y_0 + Y_1\, t$, the foregoing equations yield

$$\begin{aligned} \frac{d^2 X_1(t)}{dt^2} &= (1+\hbar)(A^2\, Y_0 + Y_1\, t), \\ \frac{d^2 X_k(t)}{dt^2} &= (1+\hbar)\frac{d^2 X_{k-1}(t)}{dt^2} + \hbar A^2\, X_{k-1}(t), \qquad k = 2, 3, \ldots \end{aligned} \tag{3.4}$$

Now, we solve the differential equation (3.4) to obtain

$$\begin{aligned} X_1(t) &= \frac{1}{6}\hbar A^2 Y_1 t^3 + \frac{1}{2}\hbar A^2 Y_0 t^2, \\ X_2(t) &= \frac{1}{120}\hbar^2 A^4 Y_1 t^5 + \frac{1}{24}\hbar^2 A^4 Y_0 t^4 + \frac{1}{6}\left(1+\hbar\right)\hbar A^2 Y_1 t^3 + \frac{1}{2}\left(1+\hbar\right)\hbar A^2 Y_0 t^2, \\ X_3(t) &= \frac{1}{5040}\hbar^3 A^6 Y_1 t^7 + \frac{1}{720}\hbar^3 A^6 Y_0 t^6 + \frac{1}{60}\left(1+\hbar\right)\hbar^2 A^4 Y_1 t^5 + \frac{1}{12}\left(1+\hbar\right)\hbar^2 A^4 Y_0 t^4 \\ &\quad + \frac{1}{6}\left(1+\hbar\right)^2 \hbar A^2 Y_1 t^3 + \frac{1}{2}\left(1+\hbar\right)^2 \hbar A^2 Y_0 t^2, \\ &\;\;\vdots \end{aligned}$$

And so on.

Setting $q = 1$ into equation (3.3), we have

$$X(t) = X_0(t) + \sum_{k=1}^{\infty} X_k(t),$$

and the approximate solution is

$$x_N(t) = X_0(t) + \sum_{k=1}^{N} X_k(t).$$

By taking the expectation to the both sides of the foregoing equation, one can have

$$E[x_N(t)] = E[X_0(t)] + \sum_{k=1}^{N} E[X_k(t)].$$

For $N = 2$ and by using the independence between Y_0, Y_1, and A, we have

$$\begin{aligned} E[x_2(t)] = E[Y_0] + E[Y_1]t + \frac{1}{6}\hbar E[A^2]E[Y_1]t^3 + \frac{1}{2}\hbar E[A^2]E[Y_0]t^2 + \frac{1}{120}\hbar^2 E[A^4]E[Y_1]t^5 + \\ \frac{1}{24}\hbar^2 E[A^4]E[Y_0]t^4 + \frac{1}{6}\left(1+\hbar\right)\hbar E[A^2]E[Y_1]t^3 + \frac{1}{2}\left(1+\hbar\right)\hbar E[A^2]E[Y_0]t^2. \end{aligned}$$

Since $A^2 \sim Be(\alpha = 2, \beta = 1)$, then we have $E[A^2] = \frac{2}{3}$, and $E[A^4] = \frac{1}{2}$. Also by using the initial conditions Y_0 and Y_1 which satisfy $E[Y_o] = 1$, $E[Y_o^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 3$, and $E[Y_0 Y_1] = 0$, we have

$$E[x_2(t)] = 1 + t + \frac{2}{9}\hbar t^3 + \frac{2}{3}\hbar t^2 + \frac{1}{240}\hbar^2 t^5 + \frac{1}{48}\hbar^2 t^4 + \frac{1}{9}\hbar^2 t^3 + \frac{1}{3}\hbar^2 t^2. \tag{3.5}$$

When $\hbar = -1$, we have

$$E[x_2(t)] = 1 + t - \frac{1}{9}t^3 - \frac{1}{3}t^2 + \frac{1}{240}t^5 + \frac{1}{48}t^4.$$

Now, we intend to find $E[(x_2(t))^2]$ as follows

First, we find $(x_2(t))^2$

$$(x_2(t))^2 = Y_0^2 + 2Y_0Y_1\,t + \frac{1}{1440}\hbar^4A^4Y_1Y_0\,t^9 + (\frac{1}{576}\hbar^4A^4Y_0^2 + \frac{1}{360}\hbar^4A^3Y_1^2 + \frac{1}{180}\hbar^3A^3Y_1^2)t^8 + \frac{1}{14400}\hbar^4A^4Y_1^2t^{10} + (\frac{4}{3}Y_0\hbar^2AY_1 + \frac{8}{3}Y_0\hbar AY_1)t^3 + (\hbar^2AY_0^2 + Y_1^2 + 2\hbar AY_0^2)t^2 + (\frac{1}{6}\hbar^4A^2Y_1Y_0 + \frac{23}{30}\hbar^2A^2Y_1Y_0 + \frac{2}{3}\hbar^3A^2Y_1Y_0)t^5 + (\frac{2}{3}Y_1^2\hbar A + \frac{1}{3}Y_1^2\hbar^2A + \frac{13}{12}\hbar^2A^2Y_0^2 + \hbar^3A^2Y_0^2 + \frac{1}{4}\hbar^4A^2Y_0^2)t^4 + (\frac{1}{45}\hbar^4A^3Y_1Y_0 + \frac{2}{45}\hbar^3A^3Y_1Y_0)t^7 + (\frac{1}{36}\hbar^4A^2Y_1^2 + \frac{1}{12}\hbar^3A^3Y_0^2 + \frac{23}{180}\hbar^2A^2Y_1^2 + \frac{1}{24}\hbar^4A^3Y_0^2 + \frac{1}{9}\hbar^3A^2Y_1^2)t^6.$$

Then, by taking the expectation to the both sides of the foregoing equation, one can have

$$E[(x_2(t))^2] = E[Y_0^2] + 2E[Y_0Y_1]t + \frac{1}{1440}\hbar^4E[A^8]E[Y_1Y_0]\,t^9 + (\frac{1}{576}\hbar^4E[A^8]E[Y_0^2] + \frac{1}{360}\hbar^4E[A^6]E[Y_1^2] + \frac{1}{180}\hbar^3E[A^6]E[Y_1^2])t^8 + \frac{1}{14400}\hbar^4E[A^8]E[Y_1^2]t^{10} + (\frac{4}{3}\hbar^2E[Y_0]E[A^2]E[Y_1] + \frac{8}{3}\hbar E[Y_0]E[A^2]E[Y_1])t^3 + (\hbar^2E[A^2]E[Y_0^2] + E[Y_1^2] + 2\hbar E[A^2]E[Y_0^2])t^2 + (\frac{1}{6}\hbar^4E[A^4]E[Y_1]E[Y_0] + \frac{23}{30}\hbar^2E[A^4]E[Y_1]E[Y_0] + \frac{2}{3}\hbar^3E[A^4]E[Y_1]E[Y_0])t^5 + (\frac{2}{3}\hbar E[Y_1^2]E[A^2] + \frac{1}{3}\hbar^2E[Y_1^2]E[A^2] + \frac{13}{12}\hbar^2E[A^4]E[Y_0^2] + \hbar^3E[A^4]E[Y_0^2] + \frac{1}{4}\hbar^4E[A^4]E[Y_0^2])t^4 + (\frac{1}{45}\hbar^4E[A^6]E[Y_1]E[Y_0] + \frac{2}{45}\hbar^3E[A^6]E[Y_1]E[Y_0])t^7 + (\frac{1}{36}\hbar^4E[A^4]E[Y_1^2] + \frac{1}{12}\hbar^3A^6E[Y_0^2] + \frac{23}{180}\hbar^2E[A^4]E[Y_1^2] + \frac{1}{24}\hbar^4E[A^6]E[Y_0^2] + \frac{1}{9}\hbar^3A^2Y_1^2)t^6.$$

Since $A^2 \sim Be(\alpha = 2, \beta = 1)$, then we have $E[A^2] = \frac{2}{3}$, $E[A^4] = \frac{1}{2}$, $E[A^6] = \frac{2}{5}$, and $E[A^8] = \frac{1}{3}$. Also by using the initial conditions Y_0 and Y_1 which satisfy $E[Y_o] = 1$, $E[Y_o^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 3$, and $E[Y_0Y_1] = 0$, we have

$$\begin{aligned} V[x_2(t)] &= E[(x_2(t))^2] - (E[x_2(t)])^2 \\ &= 1 + 2t^2 + \frac{4}{3}\hbar t^2 + \frac{13}{2700}\hbar^3t^8 + \frac{253}{1620}\hbar^3t^6 - \frac{2}{135}\hbar^3t^7 - \frac{8}{27}\hbar^3t^5 - \frac{16}{9}\hbar t^3 \\ &\quad - \frac{8}{9}\hbar^2t^3 - \frac{187}{540}\hbar^2t^5 - \frac{1}{5760}\hbar^4t^9 - \frac{1}{135}\hbar^4t^7 - \frac{2}{27}\hbar^4t^5 + \frac{2}{3}\hbar^2t^2 + \frac{5}{9}\hbar^3t^4 \\ &\quad + \frac{541}{172800}\hbar^4t^8 + \frac{79}{1620}\hbar^4t^6 + \frac{5}{36}\hbar^4t^4 + \frac{8}{9}\hbar t^4 + \frac{25}{24}\hbar^2t^4 + \frac{217}{1620}\hbar^2t^6 + \frac{1}{19200}\hbar^4t^{10} - 2t. \end{aligned}$$

When $\hbar = -1$, we have

$$V[x_2(t)] = 1 + \frac{4}{3}t^2 - \frac{19}{72}t^4 + \frac{43}{1620}t^6 - \frac{97}{57600}t^8 + \frac{1}{19200}t^{10} + \frac{8}{9}t^3 - \frac{67}{540}t^5 + \frac{1}{135}t^7 - 2t - \frac{1}{5760}t^9,$$

and when $N = 18$, we have

$$E[x_{18}] = 1 + t - \frac{1}{3}t^2 - \frac{1}{9}t^3 + \frac{1}{48}t^4 + \frac{1}{240}t^5 - \frac{1}{1800}t^6 - \frac{1}{12600}t^7 + \frac{1}{120960}t^8 + \frac{1}{1088640}t^9 + \cdots,$$

and

$$V[x_{18}] = 1 - 2t + \frac{4}{3}t^2 + \frac{8}{9}t^3 - \frac{19}{72}t^4 - \frac{67}{540}t^5 + \frac{103}{4050}t^6 + \frac{41}{4725}t^7 - \frac{247}{172800}t^8 - \frac{401}{1088640}t^9 + \cdots$$

The FINDAPR program described and listed in the Appendix was written to obtain the expectation and the variance of the approximate solutions of this example.

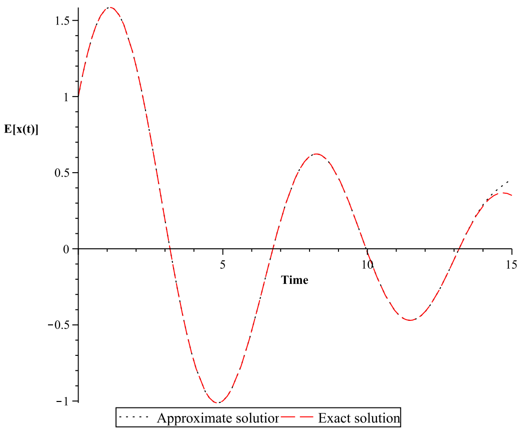

Fig. 1. Expectation approximation solution for the HAM with n = 18. For $A^2 \sim Be(\alpha = 2, \beta = 1)$

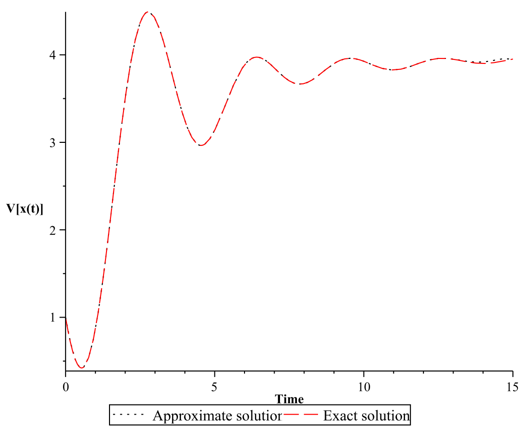

Fig. 2. Variance approximation solution for the HAM with n = 18. For $A^2 \sim Be(\alpha = 2, \beta = 1)$

3.2 Example 2

Consider the random initial value problem

$$\frac{d^2X(t)}{dt^2} + AtX(t) = 0, \quad X(0) = Y_0, \quad \frac{dX(t)}{dt}\Big|_{t=0} = Y_1, \tag{3.6}$$

where

A is a Beta random variable, $A \sim Be(\alpha = 2, \beta = 3)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 2$, $E[Y_1^2] = 5$.

we follow a similar process to that of example 3.1 to obtain

$$E[x_{18}] = 1 + 2t - \frac{1}{15}t^3 - \frac{1}{15}t^4 + \frac{1}{900}t^6 + \frac{1}{1260}t^7 - \frac{1}{113400}t^9 - \frac{1}{198450}t^{10} + \cdots$$

$$V[x_{18}] = 1 + t^2 - \frac{2}{15}t^3 - \frac{1}{15}t^5 + \frac{2}{225}t^6 + \frac{1}{450}t^7 + \frac{83}{25200}t^8 - \frac{83}{283500}t^9 - \cdots$$

3.3 Example 3

Consider random the initial value problem

$$\frac{d^2X(t)}{dt^2} + 2A\frac{dX(t)}{dt} + A^2\,X(t) = 0, \quad X(0) = Y_0, \quad \frac{dX(t)}{dt}\Big|_{t=0} = Y_1, \tag{3.7}$$

where

A is a Beta random variable, $A \sim Be(\alpha = 2, \beta = 1)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$, $E[Y_1^2] = 1$.

we follow a similar process to that of example 3.1 to obtain

$$E[x_{18}] = 1 + t - \frac{11}{12}t^2 + \frac{23}{60}t^3 - \frac{13}{120}t^4 + \frac{59}{2520}t^5 - \frac{83}{20160}t^6 + \frac{37}{60480}t^7 - \frac{143}{1814400}t^8 + \cdots$$

$$V[x_{18}] = 1 - \frac{1}{2}t^2 + \frac{4}{15}t^3 + \frac{103}{720}t^4 - \frac{649}{2520}t^5 + \frac{8959}{50400}t^6 - \frac{3133}{37800}t^7 + \frac{517}{17280}t^8 - \frac{5923}{665280}t^9 + \cdots$$

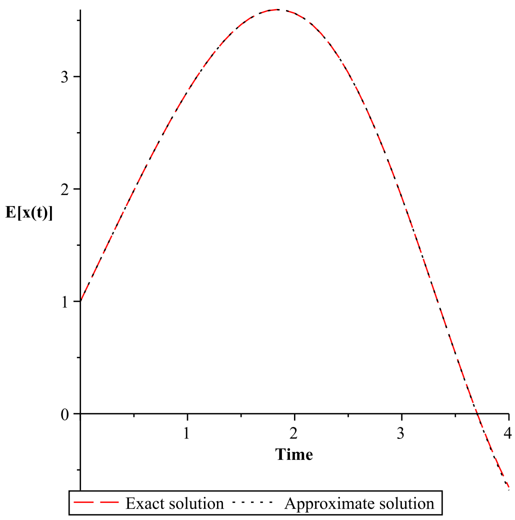

Fig. 3. Expectation approximation solution for the HAM with n = 18. For $A \sim Be(\alpha = 2, \beta = 3)$

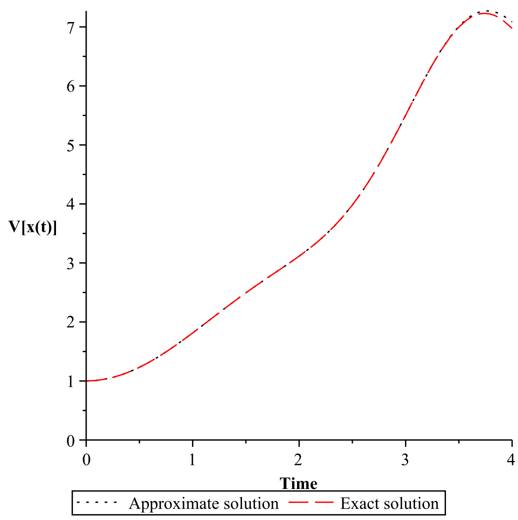

Fig. 4. Variance approximation solution for the HAM with n = 18. For $A \sim Be(\alpha = 2, \beta = 3)$

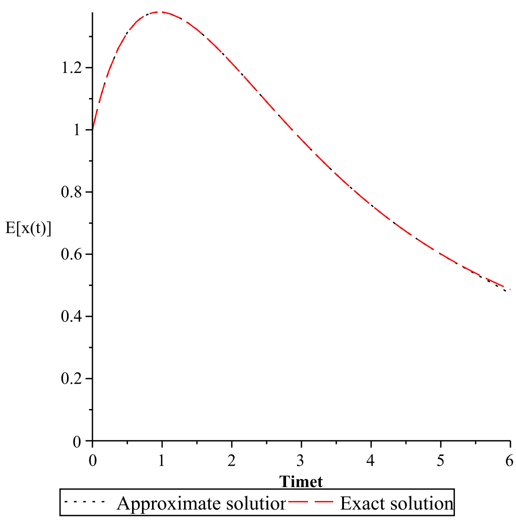

Fig. 5. Expectation approximation solution for the HAM with n = 18. For $A \sim Be(\alpha = 2, \beta = 1)$

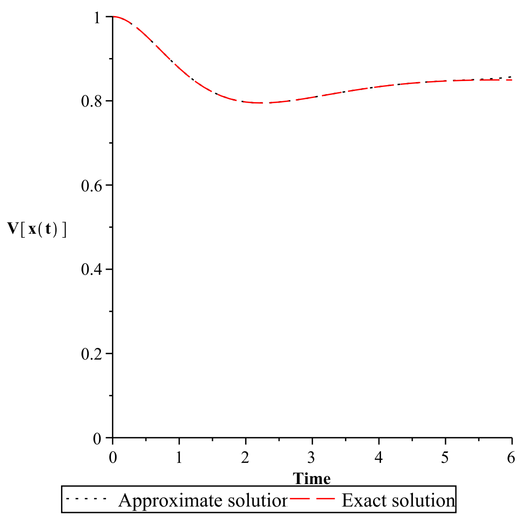

Fig. 6. Variance approximation solution for the HAM with n = 18. For $A \sim Be(\alpha = 2, \beta = 1)$

3.4 Example 4

Consider the random initial value problem

$$\frac{d^2X(t)}{dt^2} + At\,X(t) = 0, \quad X(0) = Y_0, \quad \frac{dX(t)}{dt}\,|_{t=0} = Y_1, \tag{3.8}$$

where

A is a Uniform random variable, $A \sim U(\alpha = 0, \beta = 1)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$,
$E[Y_1^2] = 1$.

we follow a similar process to that of example 3.1 to obtain

$$E[x_{18}] = 1 + t - \frac{1}{12}t^3 + \frac{1}{120}t^5 - \frac{1}{1344}t^7 + \frac{1}{17280}t^9 - \frac{1}{253440}t^{11} + \frac{1}{4193280}t^{13} - \cdots$$

$$V[x_{18}] = 1 + \frac{1}{432}t^6 - \frac{1}{1440}t^8 + \frac{131}{1008000}t^{10} - \frac{41}{2177280}t^{12} + \frac{71471}{31294771200}t^{14} - \cdots$$

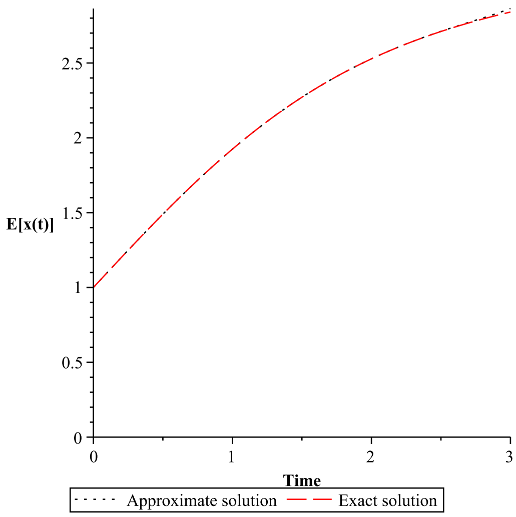

Fig. 7. Expectation approximation solution for the HAM with n = 18. For $A \sim U(\alpha = 0, \beta = 1)$

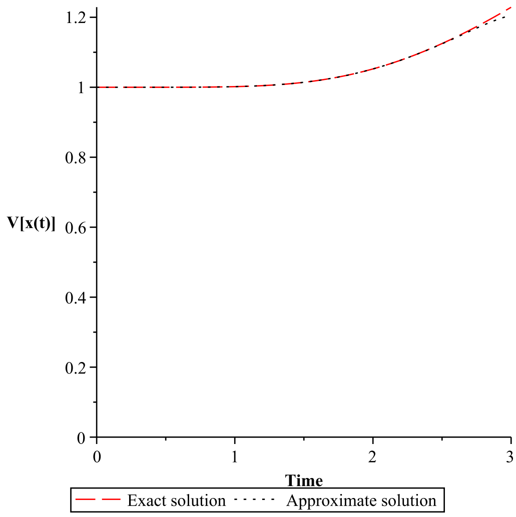

Fig. 8. Variance approximation solution for the HAM with n = 18. For $A \sim U(\alpha = 0, \beta = 1)$

3.5 Example 5

Consider the random initial value problem

$$\frac{d^2X(t)}{dt^2} + At\,X(t) = 0, \quad X(0) = Y_0, \quad \frac{dX(t)}{dt}\,|_{t=0} = Y_1, \tag{3.9}$$

where

A is a Uniform random variable, $A \sim U(\alpha = 0, \beta = 2)$, and independently of the initial conditions

Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 4$, $E[Y_1] = 1$,
$E[Y_1^2] = 2$.

we follow a similar process to that of example 3.1 to obtain

$$E[x_{18}] = 1 + t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{18}t^4 + \frac{1}{90}t^5 - \frac{1}{360}t^6 - \frac{1}{2520}t^7 + \frac{1}{12600}t^8 + \frac{1}{113400}t^9 + \cdots$$

$$V[x_{18}] = 3 - 2t^2 + \frac{13}{12}t^4 + \frac{1}{18}t^5 - \frac{61}{270}t^6 - \frac{2}{135}t^7 + \frac{599}{22680}t^8 + \frac{101}{56700}t^9 - \frac{47}{23625}t^{10} - \cdots$$

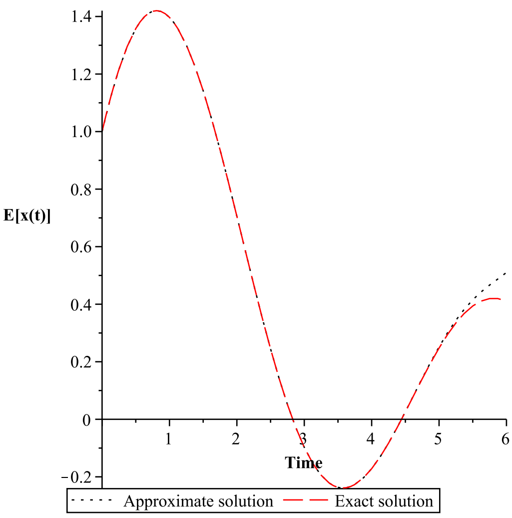

Fig. 9. Expectation approximation solution for the HAM with n = 18. For $A \sim U(\alpha = 0, \beta = 2)$

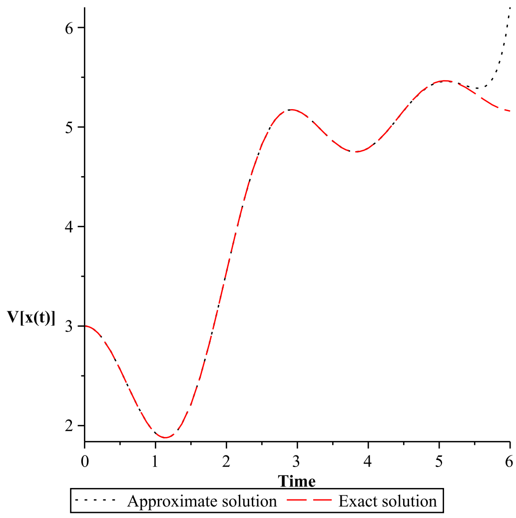

Fig. 10. Variance approximation solution for the HAM with n = 18. For $A \sim U(\alpha = 0, \beta = 2)$

3.6 Example 6

Consider the random initial value problem

$$\frac{d^2X(t)}{dt^2} + A\,X(t) = -X(t) + \sin(t), \quad X(0) = Y_0, \quad \frac{dX(t)}{dt}\,|_{t=0} = Y_1, \tag{3.10}$$

where

A is a Uniform random variable, $A \sim U(\alpha = 1, \beta = 2)$, and independently of the initial conditions Y_0 and Y_1 which satisfy $E[Y_0] = 1$, $E[Y_0^2] = 2$, $E[Y_1] = 1$,
$E[Y_1^2] = 6, E[Y_0Y_1] = 0$.

we follow a similar process to that of example 3.1 to obtain

$$E[x_{18}] = 1 + t - \frac{5}{4}t^2 - \frac{1}{4}t^3 + \frac{19}{72}t^4 + \frac{17}{720}t^5 - \frac{13}{576}t^6 - \frac{11}{8640}t^7 + \frac{211}{201600}t^8 + \cdots$$

$$V[x_{18})] = 1 - 2t + \frac{5}{2}t^2 + \frac{10}{3}t^3 - \frac{293}{144}t^4 - \frac{67}{40}t^5 + \frac{185}{288}t^6 + \frac{12221}{30240}t^7 - \frac{198419}{1814400}t^8 - \cdots$$

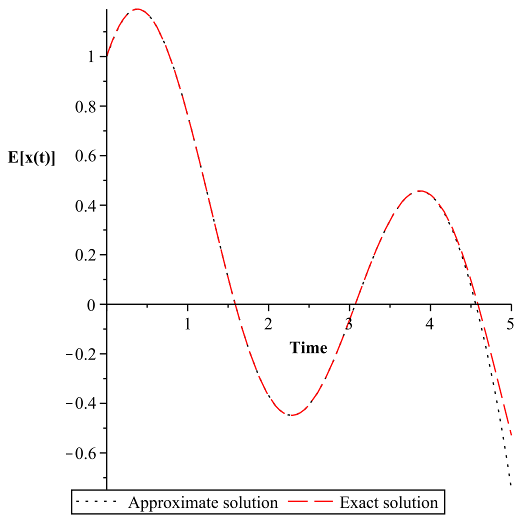

Fig. 11. Expectation approximation solution for the HAM with n = 18. For $A \sim U(\alpha = 1, \beta = 2)$

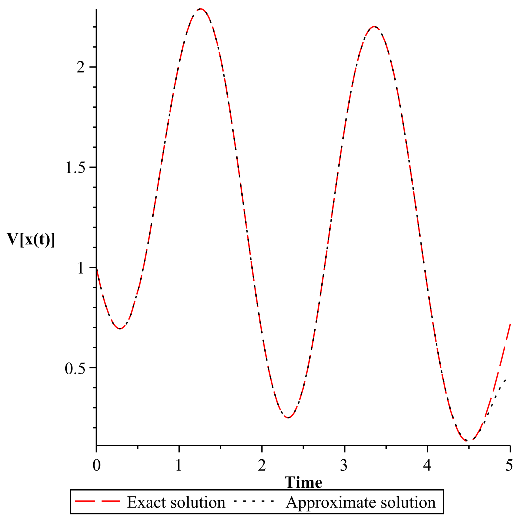

Fig. 12. Variance approximation solution for the HAM with n = 18. For $A \sim U(\alpha = 1, \beta = 2)$

4 Conclusion

The main aim of this article was to improve the work of Golmankhaneh et al. [1]. In particular equation (10) in [1] which is used to compute the the expectation approximation and the variance approximation of second-order random differential equations. It may be argued from the Fig. 4 and Fig. 6 in [1], that there was an error in obtaining the variance approximation solutions using HAM in [1]. Also, the rest of the results in [1] are not correct (the variance of the approximate solutions is negative) compared with that found by Khalaf [2], Khudair et al. [3], and Khudair et al. [4]. In this article, the homotopy analysis method is employed to obtain the expectation and the variance of the approximate solutions involve the computation of the main statistical functions of the mean square. This calculation is built in Maple software. Further, for all of the discussed examples, the expectation approximation $E[X(t)]$ plotted against the time in the Figs. 1, 3, 5, 7, 9, 11 and the variance approximation $V[X(t)]$ plotted against the time in the Figs. 2, 4, 6, 8, 10, 12. It has been shown that from Figs. 1-12 that the results for the second-order random differential equations are qualitatively exactly the same as those of Khalaf [2], Khudair et al. [3], and Khudair et al. [4].

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Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Appendix - The program FINDAPR

```
restart :
with(Statistics) :
N := 18;
A := Random Variable(('Beta')(2, 1)) :
x[0] := Y[0] + Y[1] * t :
h := −1;
for k from 0 to N do
x[k + 1] := int(int((1 + h) * diff(eval(x[k], t = s), s, s) + h * A * eval(x[k], t = s), s = 0..r), r = 0..t);
od :
for k from 0 to N do
Ex[k] := ExpectedValue(x[k]);
od :
expacted := sum(Ex[v], v = 0..N);
va := 0 :
for r from 0 to N do
for s from 0 to N do
variance := va + ExpectedValue(x[s] * x[r]);
va := variance;
od :
od :
y0 := coeftayl(variance, [Y[0], Y[1]] = [0, 0], [1, 0]) :
y1 := coeftayl(variance, [Y[0], Y[1]] = [0, 0], [0, 1]) :
y0y1 := coeftayl(variance, [Y[0], Y[1]] = [0, 0], [1, 1]) :
y00 := coeftayl(variance, [Y[0], Y[1]] = [0, 0], [2, 0]) :
y11 := coeftayl(variance, [Y[0], Y[1]] = [0, 0], [0, 2]) :
vari := ((y0 * Y[0] + y1 * Y[1] + y00 * YY[0] + y11 * YY[1])) :
YY[0] := 2 :
YY[1] := 3 :
Y[0] := 1 :
Y[1] := 1 :
varia := varia − (expacted)^2;
with(numapprox) :
with(plots) :
Exact_Ex := −(4 * (2 * Y[1] * t − 6 * Y[0] + 6 * Y[0] * cos(t) + Y[1] * t^3 * cos(t)
        −3 * Y[0] * t^2 * cos(t) − 2 * Y[1] * t * cos(t) − 2 * Y[1] * t^2 * sin(t)
```

$+6*\mathrm{Y}[0]*\mathrm{t}*\sin(\mathrm{t})-\mathrm{Y}[0]*\mathrm{t}^3*\sin(\mathrm{t})))/\mathrm{t}^4:$

$\mathrm{Exact_{Vx}} := 3\mathrm{t}^4+4-6*\cos(\mathrm{t})*\sin(\mathrm{t})\mathrm{t}^3-2*\cos(\mathrm{t})*\sin(\mathrm{t})\mathrm{t}+3*\cos(\mathrm{t})^2\mathrm{t}^2$
$-3*\cos(\mathrm{t})^2/\mathrm{t}^4-16*(2*\mathrm{t}-6+6*\cos(\mathrm{t})+\mathrm{t}^3*\cos(\mathrm{t})$
$-3*\mathrm{t}^2*\cos(\mathrm{t})-2*\mathrm{t}*\cos(\mathrm{t})-2*\mathrm{t}^2*\sin(\mathrm{t})+6*\mathrm{t}*\sin(\mathrm{t})-\mathrm{t}^3*\sin(\mathrm{t}))^2/\mathrm{t}^8:$

$\mathrm{G1} := \mathrm{plot}(\mathrm{Exact_{Ex}}, \mathrm{t}=0..15, \mathrm{linestyle}=\mathrm{dash},\ \mathrm{color}=\mathrm{red},\ \mathrm{legend}=\mathrm{Exact\ solution},$
$labels = [Time, E[x(t)]]):$

$\mathrm{F1} := \mathrm{plot}(\mathrm{expacted}, \mathrm{t}=0..15, \mathrm{linestyle}=\mathrm{dot},\ \mathrm{color}=\mathrm{black},\ \mathrm{legend}=\mathrm{Approximate\ solution},$
$labels = [Time, E[x(t)]]):$

$\mathrm{display}(\{\mathrm{F1},\mathrm{G1}\});$

$\mathrm{G} := \mathrm{plot}(\mathrm{Exact_{Vx}}, \mathrm{t}=0..15, \mathrm{linestyle}=\mathrm{dash},\ \mathrm{color}=\mathrm{red},\ \mathrm{legend}=\mathrm{Exact\ solution},$
$labels = [Time, V[x(t)]]):$

$\mathrm{F} := \mathrm{plot}(\mathrm{varia}, \mathrm{t}=0..15, \mathrm{linestyle}=\mathrm{dot},\ \mathrm{color}=\mathrm{black},\ \mathrm{legend}=\mathrm{Approximate\ solution},$
labels = [Time, V[x(t)]]):

display({ F, G});

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