Some New Three Step Iterative Methods for Solving Nonlinear Equation Using Steffensen’s and Halley Method

Najmuddin Ahmad1* and Vimal Pratap Singh1

1Department of Mathematics, Integral University, Lucknow, Uttar Pradesh, India.

Authors’ contributions
This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information
DOI: 10.9734/BJMCS/2016/29322

Abstract
In this paper, we introduce the comparative study of new three step iterative methods for finding the zeros of the nonlinear equation \( f(x) = 0 \). The new method based on the Steffensen’s method and Halley method with using predictor – corrector technique. It is established that the new method (NTSM-1) has convergence order sixth and second new method (NTSM-2) has convergence order seventh. Numerical tests show that the new methods is comparable with the well known existing methods and gives better results.

Keywords: Non linear equations; iterative methods; three step; convergence analysis; Halley method; Steffensen’s method.

1 Introduction

Numerical analysis is the area of mathematics and computer sciences that creates, analyzes and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real – world applications of algebra, geometry and calculus and they involve variables which vary continuously: These problems occur throughout the natural sciences, social sciences, engineering, medicine and business. New three step iterative methods for finding the approximate solutions of the
nonlinear equation \( f(x) = 0 \) are being developed using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [1-15] and the references therein. The most famous of these methods is the classical Newton’s method (NM) [16].

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}
\]

The Newton’s method (1) was modified by Steffensen’s method who replaced the first derivative \( f'(x) \) in Newton’s method by forward difference approximation [16].

\[
f'(x) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}
\]

and obtained the famous Steffensen’s method (SM) [17,8,16].

\[
x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)} \tag{2}
\]

Newton’s method and Steffensen’s method are of second order converges.

For a given \( x_0 \), compute approximation solution \( x_{n+1} \) by the iterative scheme

\[
x_{n+1} = x_n - \frac{2f(x_n) f'(x_n)}{2f^2(x_n) - f'(x_n) f''(x_n)} \tag{3}
\]

This is known as Halley’s method has cubic convergence (see [7,12]).

We use Predictor – corrector methods, we shall now discuss the application of the explicit and implicit multistep methods, for the solution of the initial value problems. We use explicit (predictor) method for predicting a value and then use the implicit (corrector) method iteratively until the convergence is obtained [20].

2 Iterative Methods

It is well known that a wide class of problems, which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equations of the type.

\[
f(x) = 0 \tag{4}
\]

Various numerical methods have been developed using the Taylor series and other techniques. In this paper, we use another series of the nonlinear function \( f(x) \) which can be obtained by using the trapezoidal rule and the Fundamental Theorem of Calculus. To be more precise, we assume that \( \alpha \) is a simple root of (4) and \( \gamma \) is an initial guess sufficiently close to \( \alpha \) now using the trapezoidal rule and fundamental theorem of calculus, one can show that the function \( f(x) \) can be approximated by the series [19].

\[
f(x) = f(\gamma) + \frac{x-\gamma}{2} [ f'(x) + f'(\gamma) ] \tag{5}
\]

where \( f'(x) \) is the differential of \( f \).

From (4) and (5), we have

\[
x = \gamma - 2 \frac{f(\gamma)}{f'(\gamma)} - (x - \gamma) \frac{f(x)}{f'(\gamma)} \tag{6}
\]
Using (6), one can suggest the following iterative method for solving the nonlinear equations (4).

For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative scheme [19].

$$x_{n+1} = x_n - 2 \frac{f'(x_n)}{f''(x_n)} - (x_{n+1} - x_n) \frac{f'(x_{n+1})}{f'(x_n)}$$

$$n = 0, 1, 2, 3, \ldots$$

(7)

we use the predictor – corrector technique. Using the Steffensen’s method as a predictor, Halley method and equation (6) as a corrector, we suggest and analyze the following iterative method for solving the nonlinear equation (4) and this is the main motivation of this note [17,18,14].

**Theorem 1:** For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative schemes. From equation (2), (3) and (6).

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n) + f(x_n) - f(x_n)}$$

$$b_n = a_n - \frac{2f(a_n) f'(a_n)}{2f^2(a_n) - f(a_n) f''(a_n)}$$

$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)} - (b_n - x_n) \frac{f'(b_n)}{f'(x_n)}$$

$$n=0, 1, 2, 3, \ldots$$

Theorem 1 is called the NEW THREE STEP METHOD -1 (NTSM-1) and has sixth order convergence.

**Theorem 2:** From equation (4) and equation (5) we can have

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(x_n) + f'(x_n)}$$

This is fixed point formulation enable us to suggest the following iterative method for solution the nonlinear equation.

**Theorem 3:** For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative schemes. From equation (2), (3) and theorem (2)

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n) + f(x_n) - f(x_n)}$$

$$b_n = a_n - \frac{2f(a_n) f'(a_n)}{2f^2(a_n) - f(a_n) f''(a_n)}$$

$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(b_n) + f'(x_n)}$$

$$n=0, 1, 2, 3, \ldots$$

Theorem 3 is called the NEW THREE STEP METHOD -2 (NTSM-2) and has seventh order convergence.
3 Convergence Analysis

Let us now discuss the convergence analysis of the above Theorem 1.

**Theorem 3.1**: let $\alpha \in I$ be a simple zero of sufficiently differential function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I, if $x_0$ is sufficiently close to $\alpha$ then the three step iterative method defined by theorem 1 sixth order convergence.

**Proof**: Let $\alpha$ be a simple zero of $f$. Than by expanding $f(x_n)$ and $f'(x_n)$ about $\alpha$ we have

\[
f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + \ldots \tag{8}\]

\[
f'(x_n) = 1 + \frac{2c_2}{c_1} e_n + \frac{3c_3}{c_1} e_n^2 + \frac{4c_4}{c_1} e_n^3 + \ldots \tag{9}\]

Where $c_k = \frac{1}{k!} f^{(k)}(\alpha)$, $k=1, 2, 3, \ldots$

and $e_n = x_n - \alpha$

From (8), we have

\[
[f(x_n)]^2 = c_1^2 e_n^2 + 2 c_1 c_2 e_n^3 + c_2^2 e_n^4 + \ldots \tag{10}\]

\[
f(x_n + f(x_n)) = c_1^2 e_n + (3c_1 c_2 + c_1^2 c_2 + 2c_2^2) e_n^2 + \ldots \tag{11}\]

From (10) and (11), we have

\[
\frac{[f(x_n)]^2}{f(x_n + f(x_n))} = e_n - \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^2 + \ldots \tag{12}\]

From (12), we have

\[
a_n = \alpha + \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^2 + \ldots \tag{13}\]

Let us set $A = a_n - \alpha$. Then the equation (13) can be re-written in the form

\[
A = \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^2 + \ldots \tag{14}\]

Now expanding $f(a_n), f'(a_n), f''(a_n)$ about $\alpha$ and using (13), we have

\[
f(a_n) = A c_1 + A^2 c_2 + A^3 c_3 + \ldots \tag{15}\]

\[
f'(a_n) = c_1 + 2A c_2 + A^2 3 c_3 + A^3 4 c_4 + \ldots \tag{16}\]

\[
f''(a_n) = 2c_2 + 6A c_3 + A^2 12 c_4 + \ldots \tag{17}\]

Combining (13) – (17), we have

\[
b_n = \alpha + (c_2^2 - c_3) A^3 \tag{18}\]
Also expanding \( f'(b_n) \) about \( \alpha \) and using (18), we have

\[
f'(b_n) = c_1 \left[ 1 + 2(c_1^2 - c_3) a^2 \frac{c_2}{c_1} + 3(c_1^2 - c_3)^2 A^6 \frac{c_3}{c_1} + \cdots \right] \tag{19}
\]

By substituting (8), (9), (18), (19) in theorem (1) and after some simple calculations, we obtain

\[
x_{n+1} = \alpha + (c_2^2 - c_3) \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^6 + \cdots \tag{20}
\]

\[
e_{n+1} = (c_2^2 - c_3) \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^6 + \cdots \tag{21}
\]

This shows that Theorem 1 is sixth order convergence.

Let us now discuss the convergence analysis of the above theorem 3.

**Theorem 3.2:** Let \( \alpha \in I \) be a simple zero of sufficiently differential function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( I \), if \( x_0 \) is sufficiently close to \( \alpha \) then the three step iterative method defined by theorem 3 seventh order convergence.

**Proof:**

From equation (19), we have

\[
f'(b_n) = 2c_1 e_n + 2c_2 e_n^2 + 2c_3 e_n^3 + 2c_4 e_n^4 + \cdots \tag{22}
\]

By substituting (8), (9), (22) in theorem (3) and after some simple calculations, we obtain

\[
x_{n+1} = \alpha + 2(c_2^2 - c_3) \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^6 + \cdots \tag{23}
\]

\[
e_{n+1} = 2(c_2^2 - c_3) \left( \frac{c_2}{c_1} + c_2 + 2 \frac{c_2^2}{c_1^2} \right) e_n^6 + \cdots \tag{24}
\]

This shows that Theorem 3 is seventh order convergence.

**4 Numerical Examples**

For comparisons, we have used Steffensen’s method [20] and three step predictor-corrector Newton-Halley method (PCNH) [8] defined respectively by

\[
x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}
\]

and

\[
w_n = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
y_n = w_n - \frac{2f(w_n) f'(w_n)}{2f'(w_n) - f(w_n) f''(w_n)}
\]

\[
x_{n+1} = y_n - \frac{2 f(y_n)}{f'(y_n)} \frac{f'(y_n)^2 - f(y_n) f''(y_n)}{2 f'(y_n)^2} \tag{25}
\]

\[
n = 0, 1, 2, \ldots
\]
All computations are performed using MATLAB. The following examples are used for numerical testing.

\[ f_1(x) = 3x - \sqrt{1 + \sin(x)} \]
\[ f_2(x) = x + \sin(x) - x^3 \]
\[ f_3(x) = e^x - 3x \]
\[ f_4(x) = \cos(x) - 3x + 1 \]
\[ f_5(x) = xe^x - 2 \]
\[ f_6(x) = x\log_{10}x - 1.2 \]
\[ f_7(x) = x - \cos(x) \]
\[ f_8(x) = 4x - e^x \]
\[ f_9(x) = \sin(x) - 1 - x^3 \]
\[ f_{10}(x) = e^x - 1.5 - \tan^{-1}(x) \]

As for the convergence criteria, it was required that the distance of two consecutive approximations \( \delta \) and also displayed is the number of iterations to approximate the zero (IT), the approximate zero \( x_n \) and the value \( f(x_n) \).

### Table 1. Numerical examples and comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>IT</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 ), ( x_0 = 0.5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>4</td>
<td>0.391846907002648</td>
<td>-4.44089209850063e-016</td>
<td>1.01410879693731e-10</td>
</tr>
<tr>
<td>PCNH</td>
<td>3</td>
<td>0.391846907002648</td>
<td>2.5038310085725e-008</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>2</td>
<td>0.391846907002648</td>
<td>7.53028660899213e-006</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>2</td>
<td>0.391846907002648</td>
<td>3.78833973964160e-006</td>
<td></td>
</tr>
<tr>
<td>( f_2 ), ( x_0 = 1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>5</td>
<td>1.31716296100603</td>
<td>9.76996261670138e-016</td>
<td>3.32861537577501e-005</td>
</tr>
<tr>
<td>PCNH</td>
<td>6</td>
<td>1.31716296100603</td>
<td>4.68398653197255e-011</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>1.31716296100603</td>
<td>5.34990427110637e-012</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>1.31716296100603</td>
<td>6.474839973964160e-006</td>
<td></td>
</tr>
<tr>
<td>( f_3 ), ( x_0 = 0.2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>4</td>
<td>0.619061286735945</td>
<td>2.22044604925031e-016</td>
<td>7.0389180040209e-009</td>
</tr>
<tr>
<td>PCNH</td>
<td>5</td>
<td>0.619061286735945</td>
<td>6.835597853691e-011</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>0.619061286735945</td>
<td>3.10652923940502e-007</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>0.619061286735945</td>
<td>3.59104859182224e-008</td>
<td></td>
</tr>
<tr>
<td>( f_4 ), ( x_0 = 0.2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>5</td>
<td>0.607101648103123</td>
<td>-1.33226762955019e-015</td>
<td>9.99200722162641e-016</td>
</tr>
<tr>
<td>PCNH</td>
<td>4</td>
<td>0.607101648103123</td>
<td>5.16754083967896e-010</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>0.607101648103123</td>
<td>6.95768997971413e-011</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>0.607101648103123</td>
<td>3.65396601864632e-012</td>
<td></td>
</tr>
<tr>
<td>Method</td>
<td>IT</td>
<td>$x_n$</td>
<td>$f(x_n)$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>--------</td>
<td>----</td>
<td>-------</td>
<td>----------</td>
<td>---------</td>
</tr>
<tr>
<td>$f_{5}, x_0 = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>6</td>
<td>0.852605502013726</td>
<td>-2.44249065417534e-015</td>
<td>3.517610647279e-010</td>
</tr>
<tr>
<td>PCNH</td>
<td>4</td>
<td>0.852605502013726</td>
<td>2.43496630636386e-008</td>
<td>2.68309041651094e-010</td>
</tr>
<tr>
<td>NTSM-1</td>
<td>4</td>
<td>0.852605502013726</td>
<td>2.88919999036352e-011</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>0.852605502013726</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_{6}, x_0 = 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>4</td>
<td>2.74064609597369</td>
<td>-3.10862446895044e-015</td>
<td>1.70161289503313e-008</td>
</tr>
<tr>
<td>PCNH</td>
<td>4</td>
<td>2.74064609597369</td>
<td>1.95032203720302e-008</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>2.74064609597369</td>
<td>3.98418986691024e-009</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>2.74064609597369</td>
<td>1.98129956885396e-010</td>
<td></td>
</tr>
<tr>
<td>$f_{7}, x_0 = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>5</td>
<td>0.739085133215161</td>
<td>6.66133814775094e-016</td>
<td>7.13984427136438e-013</td>
</tr>
<tr>
<td>PCNH</td>
<td>4</td>
<td>0.739085133215161</td>
<td>7.05561942204724e-010</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>0.739085133215161</td>
<td>4.489976282798e-011</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>0.739085133215161</td>
<td>2.4510393716496e-012</td>
<td></td>
</tr>
<tr>
<td>$f_{8}, x_0 = 1.8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>6</td>
<td>2.15329236411035</td>
<td>-1.77635683940025e-015</td>
<td>5.27400345617934e-011</td>
</tr>
<tr>
<td>PCNH</td>
<td>Divergent</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>2.15329236411035</td>
<td>1.4824609840321e-008</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>2.15329236411035</td>
<td>2.2986798486234e-008</td>
<td></td>
</tr>
<tr>
<td>$f_{9}, x_0 = -1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>4</td>
<td>-1.24905214850119</td>
<td>-1.9984014432528e-014</td>
<td>0.000227912085001725</td>
</tr>
<tr>
<td>PCNH</td>
<td>5</td>
<td>-1.24905214850119</td>
<td>2.5383993469285e-009</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>-1.24905214850119</td>
<td>1.20446399520802e-008</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>-1.24905214850119</td>
<td>1.01799013663140e-009</td>
<td></td>
</tr>
<tr>
<td>$f_{10}, x_0 = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>5</td>
<td>0.767653266201279</td>
<td>0</td>
<td>2.27928786955545e-012</td>
</tr>
<tr>
<td>PCNH</td>
<td>5</td>
<td>0.767653266201279</td>
<td>1.48392409471398e-012</td>
<td></td>
</tr>
<tr>
<td>NTSM-1</td>
<td>3</td>
<td>0.767653266201279</td>
<td>9.48948919443637e-010</td>
<td></td>
</tr>
<tr>
<td>NTSM-2</td>
<td>3</td>
<td>0.767653266201279</td>
<td>1.48009937639415e-010</td>
<td></td>
</tr>
</tbody>
</table>

5 Conclusion

In this paper, we have suggested and analyzed newly developed technique is faster than Steffensen’s method (SM) and predictor-corrector Newton-Halley method (PCNH). This method based on a Steffensen’s method and Halley method and using predictor – corrector technique. Our method can be considered as significant improvement of Steffensen’s method and PCNH and can be considered as alternative method of solving nonlinear equations.

Competing Interests

Authors have declared that no competing interests exist.

References


ISSN 2347-8527


DOI: 10.1137/1.9780898719468


ISSN: 1314-3395 (on-line version)


DOI: 10.1080/00207160.2010.520705


DOI: 10.1007/s11075-013-9715-x


(Article in press)

DOI: 10.1007/s12190-013-0647-3

© 2016 Ahmad and Singh; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/16602