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Ostrowski Type Inequalities for Functions whose Derivatives are H-convex Via Fractional Integrals

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Author's contribution

This whole work was carried out by the author MM.

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ABSTRACT

In this paper we have established some Ostrowski type inequalities involving Riemann – Liouville fractional integrals for functions whose derivatives are h-convex.

Keywords: Riemann – liouville integrals; ostrowski type inequalities; convex function; s-convex function; h-convex function.

1. INTRODUCTION

If $f: I \subseteq [0, \infty) \to R$ be a differentiable mapping on I°, the interior of the interval *I*, such that $f' \in L_1([a, b])$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \mathrm{d}u \right| \leq \frac{M}{b-a} \cdot \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \qquad (1.1)$$

holds. This results is known in the literature as the Ostrowski inequality. For recent results and generalizations concerning Ostrowski's inequality see [1-6] and the references therein.

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The following definition is well known in the literature:

a function $f: I \to R,\ \emptyset \neq I \subseteq R$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.2)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In 1978, Breckner [7] introduced an *s*-convex function as a generalization of a convex function. Such a function is defined in the following way: A function $f: [0,\infty) \to R$, is said to be *s*-convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \quad (1.3)$$

holds for all $x, y \in [0,\infty),\ t \in [0,1]$ and for fixed $s \in [0,1]$.

In 2007, Varošanec [8] introduced a large class of non-negative functions, the so-called *h*-convex functions. This class contains several well-known classes of functions such as non-negative convex functions. This class is defined in the following way: a non-negative function $f: I \to R,\ \emptyset \neq I \subseteq R$, is an interval, is called *h*-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad (1.4)$$

holds for all $x, y \in I$ and $t \in [0,1]$, where $h: J \to R$ is a non-negative function, $h \not\equiv 0$ and J is an interval, $(0,1) \subseteq J$.

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used futher in this paper. For more details, one can consult [9-11].

Let $f \in L_1([a,b])$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \mathrm{d}t \quad , \quad x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \mathrm{d}t \quad , \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

The aim of this paper is to establish Ostrowski type inequalities involving Riemann-Liouville fractional integrals for functions whose derivatives are *h*-convex.

During the reviewing process it turned out that similar results had been obtained by Liu in [12] but under the additional assumption that h is super-additive or super-multiplicative function. This means that the class of the considered functions were limited. For example if h is a super-additive function then we remove from our consideration the s-convex function.

2. OSTROWSKI TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

We need the following lemma which results from [13] Lemma 2 proof.

Lemma 1. Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$. If $f^{'}\in L_1([a,b])$, then for all $x\in(a,b)$ the following equality for fractional integrals holds:

$$f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]$$

$$=\frac{x-a}{2}\int_0^1 t^{\alpha}f^{'}(tx+(1-t)a)\mathrm{d}t-\frac{(b-x)}{2}\int_0^1 t^{\alpha}f^{'}(tx+(1-t)b)\mathrm{d}t. \qquad (2.1)$$

Using the Lemma 1, we can obtain the following fractional integral inequalities.

Theorem 1. Let $f:I\subseteq[0,\ \infty)\to R$ be a differentiable mapping on I° such that $f^{'}\in L_1([a,b])$, where $a,b\in I$, with $a<b$. If $|f'|$ is *h*-convex on $[a,b]$ and $|f'(x)|\leq M(M>0)$, $x\in[a,b]$, then the following inequality holds:

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{M(b-a)}{2}\int_0^1 t^{\alpha}\,[h(t)+h(1-t)]\mathrm{d}t, \qquad (2.2)$$

for each $x\in[a,b]$.

Proof. By Lemma 1 and since $|f'|$ is *h*-convex, then we have

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{x-a}{2}\int_0^1 t^{\alpha}\left|f^{'}(tx+(1-t)a)\right|\mathrm{d}t+\frac{b-x}{2}\int_0^1 t^{\alpha}\left|f^{'}(tx+(1-t)b)\right|\mathrm{d}t$$

$$\leq \frac{x-a}{2}\int_0^1 t^{\alpha}\left[h(t)\left|f'(x)\right| + h(1-t)\left|f'(a)\right|\right]\mathrm{d}t$$

$$+\frac{b-x}{2}\int_0^1 t^{\alpha}\left[h(t)\left|f'(x)\right| + h(1-t)\left|f'(b)\right|\right]\mathrm{d}t$$

$$\leq M\frac{x-a}{2}\int_0^1 t^{\alpha}\left[h(t)+h(1-t)\right]\mathrm{d}t + M\frac{b-x}{2}\int_0^1 t^{\alpha}\left[h(t)+h(1-t)\right]\mathrm{d}t$$

$$=\frac{M(b-a)}{2}\int_0^1 t^{\alpha}\left[h(t)+h(1-t)\right]\mathrm{d}t,$$

what completes the proof.

Corollary 1. In Theorem 1, if $|f'|$ is convex, then we get the following inequality

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right| \leq \frac{M(b-a)}{2(\alpha+1)}. \quad (2.3)$$

Corollary 2. In Theorem 1,if we take $h(t)=t^s$, which means that $|f'|$ is *s*-convex, then inequality (2.2) becomes the following inequality

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{M(b-a)}{2}\left[\frac{1}{\alpha+s+1}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]. \quad (2.4)$$

Corollary 3. In Theorem 1, if we take$x=\frac{a+b}{2}$ and $\alpha=1$, then we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M(b-a)}{2}\int_0^1 t\left[h(t)+h(1-t)\right]\mathrm{d}t. \quad (2.5)$$

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 2. Let $f: I \subseteq [0, \infty) \to R$ be a differentiable mapping on I° such that $f^{'} \in L_1([a,b])$, where $a, b \in I$, with $a < b$. If $|f^{'}|^q$ is *h*-convex on $[a,b]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M (M > 0)$, $x \in [a,b]$, then the following inequality holds:

$$\left| f(x) - \Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}} J_{x^-}^{\alpha} f(a) + \frac{1}{2(b-x)^{\alpha}} J_{x^+}^{\alpha} f(b)\right]\right| \leq \frac{M(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(2\int_0^1 h(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \quad (2.6)$$

for each $x \in [a,b]$.

Proof. From Lemma 1 and using the Hőlder's integrals inequality, we have

$$\left| f(x) - \Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}} J_{x^-}^{\alpha} f(a) + \frac{1}{2(b-x)^{\alpha}} J_{x^+}^{\alpha} f(b)\right]\right|$$

$$\leq \frac{x-a}{2}\int_0^1 t^{\alpha}\left|f^{'}(tx+(1-t)a)\right|\mathrm{d}t + \frac{b-x}{2}\int_0^1 t^{\alpha}\left|f^{'}(tx+(1-t)b)\right|\mathrm{d}t$$

$$\leq \frac{x-a}{2}\left(\int_0^1 t^{\alpha p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)a|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{b-x}{2}\left(\int_0^1 t^{\alpha p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+(1-t)b|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{x-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\int_0^1 \left[h(t)\left|f^{'}(x)\right|^q + h(1-t)\left|f^{'}(a)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{b-x}{2(\alpha p+1)^{\frac{1}{p}}}\left(\int_0^1 \left[h(t)\left|f^{'}(x)\right|^q + h(1-t)\left|f^{'}(b)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$= \frac{M(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(2\int_0^1 h(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

This completes the proof.

Corollary 4. In Theorem 2, if $\left|f'\right|^q$ is convex then we get the following inequality

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^\alpha}J_{x^-}^\alpha f(a)+\frac{1}{2(b-x)^\alpha}J_{x^+}^\alpha f(b)\right]\right|$$

$$\leq \frac{M(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}. \tag{2.7}$$

Corollary 5. In Theorem 2, if $\left|f'\right|^q$ is *s*-convex, then inequality (2.6) becomes the following inequality

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^\alpha}J_{x^-}^\alpha f(a)+\frac{1}{2(b-x)^\alpha}J_{x^+}^\alpha f(b)\right]\right|$$

$$\leq \frac{M(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}. \tag{2.8}$$

Corollary 6. In Theorem 2, if we take$x=\frac{a+b}{2}$ and $\alpha=1$, thenwe get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leq \frac{M(b-a)}{2(p+1)^{\frac{1}{p}}}\left(2\int_0^1 h(t)\,dt\right)^{\frac{1}{q}}. \tag{2.9}$$

Theorem 3. Let $f: I\subseteq[0,\ \infty)\to R$ be a differentiable mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I$, with$a<b$. If $\left|f'\right|^q$ is *h*-convex on $[a,b]$,$q\geq 1$, and $|f'(x)|\leq M(M>0)$, $x\in[a,b]$, then the following inequality holds:

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^\alpha}J_{x^-}^\alpha f(a)+\frac{1}{2(b-x)^\alpha}J_{x^+}^\alpha f(b)\right]\right|$$

$$\leq \frac{M(b-a)}{2}\cdot\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha\,[h(t)+h(1-t)]\mathrm{d}t\right)^{\frac{1}{q}} \tag{2.10}$$

for each $x \in [a,b]$.

Proof. From Lemma 1 and using the Hőlder's integrals inequality, we have

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq \frac{x-a}{2}\int_0^1 t^{\alpha}\left|f'(t+(1-t)a)\right|\mathrm{d}t+\frac{b-x}{2}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|\mathrm{d}t$$

$$\leq \frac{x-a}{2}\left(\int_0^1 t^{\alpha}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(tx+(1-t)a|^{q}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{b-x}{2}\left(\int_0^1 t^{\alpha}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(tx+(1-t)b|^{q}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{x-a}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left[h(t)\left|f'(x)\right|^{q}+h(1-t)\left|f'(a)\right|^{q}\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{b-x}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left[h(t)\left|f'(x)\right|^{q}+h(1-t)\left|f'(b)\right|^{q}\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{M(x-a)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}[h(t)+h(1-t)]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{M(b-x)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}[h(t)+h(1-t)]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$= \frac{M(b-a)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}[h(t)+h(1-t)]\mathrm{d}t\right)^{\frac{1}{q}},$$

what completes the proof.

Corollary 7. In Theorem 3, if $\left|f'\right|^q$ is convex, then we get the following inequality

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right| \le \frac{M(b-a)}{2}\left(\frac{1}{\alpha+1}\right). \quad (2.11)$$

Corollary 8. In Theorem 3, if $\left|f'\right|^q$ is *s*-convex, then inequality (2.11) becomes the following inequality

$$\left|f(x)-\Gamma(\alpha+1)\left[\frac{1}{2(x-a)^{\alpha}}J_{x^-}^{\alpha}f(a)+\frac{1}{2(b-x)^{\alpha}}J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\le \frac{M(b-a)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{1}{\alpha+s+1}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]^{\frac{1}{q}}. \quad (2.12)$$

Corollary 9. In Theorem 3, if we take $x=\frac{a+b}{2}$ and $\alpha=1$, then we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{M(b-a)}{2^{2-\frac{1}{q}}}\left(\int_0^1 t^{\alpha}[h(t)+h(1-t)]\mathrm{d}t\right)^{\frac{1}{q}}. \quad (2.13)$$

3. CONCLUSION

In this paper we proved the Ostrowski type inequalities for functions whose derivatives are h-convex and we pointed out the results for some special classes of functions.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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